IL
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Cent'anni...

... sono passati da quando, proprio sul finire del 1854, Carlo Matteucci, fisico, e Raffaele Piria, chimico, professori della Università di Pisa già saliti in chiara fama, decidevano, sostenuti dalla fiducia e dalla fede che ispira la buona causa, di riprendere la pubblicazione di quel giornale, Il Cimento, *che essi stessi insieme con Ottaviano Fabrizio Mossotti, Luigi Pacinotti, Leopoldo Pilla e i due Savi, Pietro e Paolo, tutti professori dell'Università di Pisa, avevano fondato e mille difficoltà finanziarie, politiche, belliche avevano portato in brevi anni alla fine.*

La rinascita si fregiò di un aggettivo che voleva indicare un rinnovamento, e il nuovo giornale si chiamò Il Nuovo Cimento.

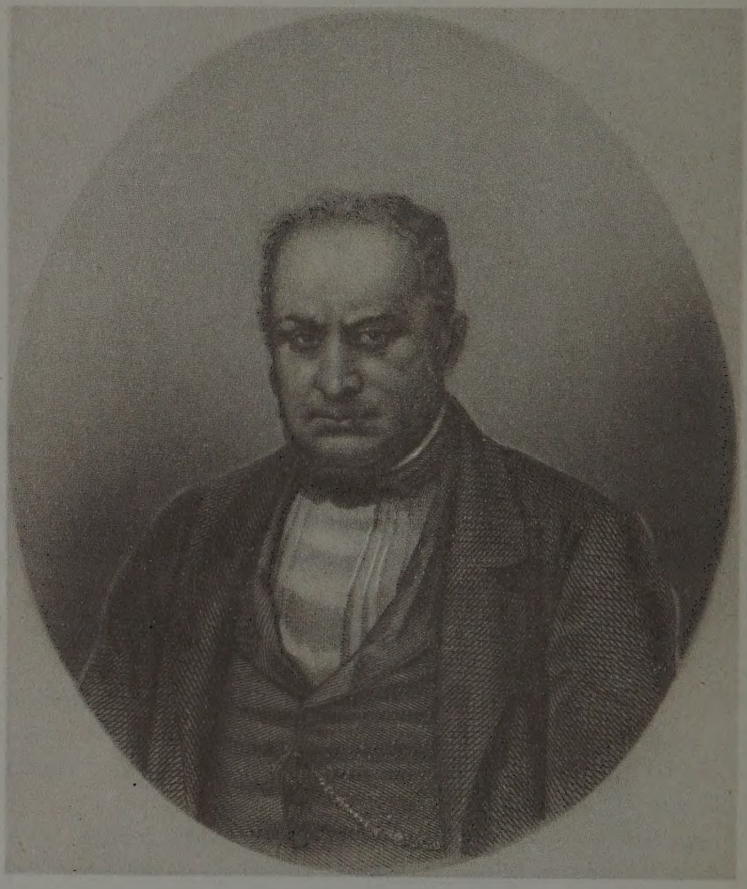
Ma il nome svelava vecchie radici, prettamente italiane: quelle che avevano nutrito la rigogliosa pianta, sorta a Firenze nel solstizio dell'anno 1657, per lo spirito curioso e novatore del Principe Leopoldo: l'Accademia del Cimento.

Ed oggi che con questo fascicolo Il Nuovo Cimento *entra nel suo centesimoprimo anno di vita, abbiamo voluto che il legame ideale con l'antica e gloriosa Accademia venisse ulteriormente rafforzato, aggiungendo sotto il titolo, in nuovo frontespizio, l'impresa di essa: il fornello acceso, i tre crogiuoli pieni di metallo fuso e il cartiglio svolazzante col motto* « provando e riprovando ».

« Cimento », nel suo pregnante significato, è, al tempo stesso, il « saggio », la « prova », lo « sforzo », il « rischio », il « pericolo », l'« esperimento », il « paragone », il « tormento » dell'indagine, la « mistura » con la quale il metallo nel crogiuolo si affina al fuoco. Il crogiuolo poi è la mente. E le due parole estreme dell'impresa, specchiandosi tra loro, indicano la via per conseguire, attraverso la prova e la riprova,

« di bella verità... il dolce aspetto ».

È l'essenza del metodo galileiano.



CARLO MATTEUCCI

★ Forlì, 21 Giugno 1811
† Ardenza (Livorno), 25 Giugno 1868

I FONDATORI D

RAFFAELE PIRIA

* Scilla di Calabria, 20 Agosto 1814
† Torino, 18 Luglio 1865

UOVO CIMENTO„

Infatti « dove non ci è più lecito metter piede innanzi, non vi à cui meglio rivolgersi, che alla fede dell'esperienza, la quale non altrimenti di chi varie gioie sciolte, e scommesse cercasse di rimettere ciascuna per ciascuna al suo incastro, così Ella adattando effetti a cagioni, e cagioni ad effetti, se non di primo lancio, come la geometria, tanto fa, che provando, e riprovando *le riesce talora di dar nel segno ».*

Così scriveva, l'anno 1667, l'anonimo autore del Proemio dei Saggi di naturali esperienze fatte nell'Accademia del Cimento.

* * *

Chi sfogli uno dopo l'altro i numerosi volumi di questo secolo di vita del nostro giornale, segue agevolmente passo passo la storia della Fisica italiana in quel periodo. E, mentre sullo scenario dei mille e mille scritti consegnati alle sue pagine vede proiettarsi di quella gli alti e i bassi, il felice episodio dell'unione nel 1897 con la nascente Società Italiana di Fisica, e quello penoso e pericoloso, del momentaneo arresto delle pubblicazioni alla fine della guerra, nel 1945 — incontro a lui si muovono in lunga teoria le figure dei nostri fisici: ecco l'Amici, ecco il Belli, il Mossotti, il Matteucci, il Felici; ecco il Betti, il Donati, il Beltrami, il Pacinotti, il Roiti; ecco il Macaluso, il Ferraris, il Righi, il Bartoli, il Calzecchi-Onesti; ecco il Volterra, il Battelli, il Garbasso, il Levi-Civita, il Puccianti, il Corbino, il Lo Surdo; ecco Ettore Majorana; ecco, ancor tiepido nel suo cenere, Enrico Fermi...; e costoro solo per nominare i maggiori tra i morti.

Ma compiuto verso il passato un atto doveroso di lode e di riconoscenza, al futuro tendiamo col nostro sguardo e la nostra volontà per affrontare e superare i numerosi doveri che oggi più che mai incombono.

Nel valicare il secolo alcuni tra i maggiori fisici hanno voluto accompagnarci: Schrödinger, Dirac, de Broglie, Hess; ne traggiamo, grati a loro, buon auspicio.

E ora avanti; in cammino, al « cimento ». ...

Milano, 25 Dicembre 1954.

G. Polvani

The Philosophy of Experiment.

E. Schrödinger

Dublin Institute for Advanced Studies - Dublin

(ricevuto il 2 Luglio 1954)

Summary. — The accepted outlook in quantum mechanics (q.m.) is based entirely on its theory of measurement. Quantitative results of observations are regarded as the only accessible reality, our only aim is to predict them as well as possible from other observations already made on the same physical system. This pattern is patently taken over from the positional astronomer, after whose grand analytical tool (analytical mechanics) q.m. itself has been modelled. But the laboratory experiment hardly ever follows the astronomical pattern. The astronomer can do nothing but observe his objects, while the physicist can interfere with his in many ways, and does so elaborately. In astronomy the time-order of *states* is not only of paramount practical interest (e.g. for navigation), but it was and is the only method of discovering the *law*, known by now in its general features (Newton). The physicist is nearly always still out for discovering the *law* (technically speaking; a Hamiltonian); this he rarely, if ever, attempts by following a single system in the time-succession of its states, which in themselves are of no interest. The accepted foundation of q.m. claims to be intimately linked with experimental science. But actually it is based on a scheme of measurement which, because it is entirely antiquated, is hardly fit to describe any relevant experiment that is actually carried out, but a host of such as are for ever confined to the imagination of their inventors.

1. – The Accepted Scheme.

In quantum theory (as used at present in thinking about experimental investigations) the following conceptions prevail: some physical *system* with which we are concerned, not necessarily isolated but possessing an individuality and more or less clearly demarcated from other parts of the physical world; the *nature* of this system and of its interaction with the surrounding, which includes

the experimenter and his measuring appliances; the *state* in which the system finds itself (some prefer to say: is found) at a given moment; *measurements* performed on the system.

The objective of physical science according to the most reserved and cautious group is to foretell what might be called the « orbit » of the state of the system, its development in time. The means both for making the prophecy and for checking it are measurements. Hence it amounts to forecasting the results of later measurements from those previously performed. Though the forecast is usually not precise but of probability, there is an unambiguous representative of the *state*, the state-vector or state-function, which is supposed to change between measurements in a precisely known fashion (if the nature of the system is known) and to determine precisely the probability forecast for any measurement at any given moment.

It is to be noted here that the terms « prophecy », « forecast », « previously », « later » must be understood to include in the limit the case of time difference zero between the two measurements, the one from which and the one for which the forecast is made. This limiting case is not trivial and not at all simple, since many different pairs of measurement can be performed on the same system in immediate succession, and not even then does the result of the first as a rule permit a unique forecast on that of the second, but only of probability.

The *nature* of the system is described by first indicating the variables on which its state-function depends and then the so-called Hamiltonian operator, which determines the partial differential equation according to which the state-function changes while undisturbed by the observer. The nature of the interaction between the system and the observer's appliances is described by a particular operator, said to be associated with any particular measuring device; it is required for making the forecast.

Except in the limiting case of time difference zero, mentioned just before, the nature of the system must be completely known, if the result of a measurement is to serve for pronouncing on the probable results of a later measurement, that is to say if it is to serve any purpose at all. For unless the Hamiltonian is known one does not know how the state of the system has changed in the meantime. There may, of course, be « constants of the motion » i.e. measuring devices for which the prediction does not change with time. They are those whose associated operator *commutes* with the Hamiltonian. But to tell whether it does the Hamiltonian must be known.

In the limiting case this knowledge is irrelevant. But this limiting case applies only to a handful of basic kinematic concepts, mostly such as played already a prominent part in dynamics ever since it exists, long before the advent of quantum theory. The prediction is in these cases based on the mutual commutation relations between the associated operators. A well known example is the cartesian coordinates of the mass centre and the components of its ve-

locity. A precise determination of one of the latter makes any value of the corresponding coordinate equally probable. More generally, the same holds for any observable parameter and its canonically conjugate; e.g. for a so-called angle variable and its corresponding action variable (in a conditionally periodic system). A different instance of considerable interest is the total angular momentum and its three cartesian components. If the former and one of the latter be determined with precision (which can be done, because their associated operators commute), then the absolute value (but not the direction) of the component orthogonal to the one that has been measured can be indicated with precision, while for the component in any particular direction in that orthogonal plane the exact probability distribution can be computed from the mutual commutation relations between the associated operators. It is not necessary to know the analytical expressions of the latter in terms of the variables on which the state function of the system depends, indeed it is not necessary to know anything about these variables, nor any details about the nature of the system. But these are exceptional cases, and positively restricted to the limit « time difference zero » (physicists dub them quantum kinematics as against quantum dynamics). In this paper we shall be concerned with the general case.

2. – The Accepted Scheme Claims Philosophical Purity.

The point I wish to make is this. The method of forecasting that I have outlined above (without the analytical details, which the physicist knows well enough, while they would bother the non-mathematician) forms the mainstay of the accepted quantum theory; it may or may not be appropriate where it applies. But whether or no it is so, we ought to consider if its claim is justified to be an accomplished theory of measurement that applies, in principle, to all cases. I hope to show by a brief anaylsis that it is very far from doing so. To say the least, the vast majority of measurements actually performed in the laboratory have an entirely different character and simply do not fall under the adopted scheme. The question whether there are any that do is comparatively of minor importance and may be touched upon later. For if only my first contention is true, it characterizes the scheme as a mere ingenious thinking-device, a scheme of the writing desk. This in itself is no degradation. Indeed the ingredients from which the great theories of the XIXth and XXth centuries were formed (Maxwell's, Gibb's, Boltzmann's Lorentz', Planck's, Einstein's) were all of this kind — pictures in the mind from which only after elaborate theoretical reasoning results, testable by experiment, can be deduced. But the present case is different. Quantum mechanics claims that it deals ultimately and directly with nothing but actual observations, since they are the only real thing, the only source of information,

which is only about *them*. The theory of measurement is carefully phrased so as to make it epistemologically unassailable. There is no question ever of what *is* or *is not* at a given instant, only of what we should *find* if we made this or that measurement; and the theory is only about the functional connexion between some group of such findings and some other group. But what is all this epistemological fuss for, if we have not to do with actual, real findings « in the flesh », only with imagined findings? And worse still, is not the whole epistemology of the scheme exploded, if there are any measurements at all, valuable sources of information, that do not fall under the scheme?

3. – The Laboratory Pattern is Different.

In the physical laboratory (as against the astronomical observatory) we are not very often interested in the future history of the body or system on which we have made a measurement. In the vast category of measurements concerned with some constant of the material (as density, compressibility, Young's modulus, specific heat, electric or thermic conductivity, surface tension, viscosity, etc.) the physical object is just a sample that may afterwards be thrown into the dustbin. The results are used on a hundred later occasions, but not usually for predicting the future behaviour of the sample. When a motion or, more generally, a change with time becomes relevant, it is more often that of a measuring instrument (the needle of a galvanometer or electrometer, the cathode ray pencil of an oscillograph) than that of the object under examination. These remarks refer not only to old fashioned routine, but also to provinces very relevant to quantum theory: blackbody-radiation, spectrometry, mass-spectrometry, nuclear magnetism, etc..

It behoves me to mention examples to the contrary: the direct determination of radioactive decay, or the observation of slow chemical reaction rates, when samples are taken and analysed from hour to hour or from day to day. The closest similarity to the scheme of quantum mechanics, to my mind, obtains in synthetic chemical manufacture of drugs. Here we actually perform some carefully prescribed preparatory operations, including a host of measurements, with the exclusive scope of producing a substance whose chemical properties we can foretell. This is a wide and important, still a very special branch of physical science. Ought one perhaps to put the manufacture of a scientific instrument on the same level? By a certain handling of raw material we produce a system — the instrument — with very special, closely predictable properties. I will not decide this at the moment and beg to regard it as a side remark.

How is it now that there are, at any rate, hosts of actual measuring devices which are continually applied and seem to fit so badly into the quantum

theory of measurement? Is that really so, or could they be looked upon at an other angle and would then fit into the scheme? No. This is really so and it is not difficult to tell the reason, and even to phrase it according to quantum mechanics' own concepts and terminology.

4. – Astronomy - The Prototype of Physical Theory.

Both forms of quantum mechanics (the matrix- and the wave-form) originated from *analytical mechanics* (a.m.). They both leaned against the great central theorems, due to Hamilton and Jacobi, of this most accomplished and highly architectural theory in physics. Let us note, by the way, that though both groups of discoverers used this architecture as a guide for initiating a new science, they did this in so entirely different ways that it was a great surprise to find them willy-nilly running into the same mathematical construct. The earlier form (HEISENBERG, BORN) led very directly to, nay it consisted in, adopting an axiom (now usually called a theorem) of dangerously fascinating beauty: the equations of motion must be taken over from a.m. *au pied de la lettre*, but the variables whose change in time they control and whose numerical values at any moment of time would in a.m. indicate the instantaneous state of the system must now be looked upon as something entirely different. They are not ordinary numbers; the product of any two of them depends in general on the order of the factors; their « commutation relations » are of outstanding importance. They are momentously *contributory* to our knowledge of the state of the system; however alone by themselves, even when completely known, they tell us absolutely nothing about the *state* (not even by probability), but only about the *nature* of the system, about *possibilities* (see Section 1 for the distinction between state and nature). That is why I called this axiom-theorem dangerously fascinating. Its apparent simplicity — the same equations of motion between quantities, habitually given the same names and represented by the same symbols — seduces us to underrate the change that has taken place. And that the more, because the analogy with a.m. goes even further. In the early stages of matrix mechanics the state function (alluded to in Section 1) was missing; it was supplied by wave-mechanics. Now, if it be given for any time, e.g. for $t=0$, then those non-commuting quantities controlled by the equations of motion do give us full information about the state at any other time. Thus the knowledge of the state-function for one moment is apparently the analogue of the initial conditions (or integration constants) in a.m.. Moreover, just as in a.m., interesting information of a general kind can be obtained from the equations of motion alone: e.g. when they assert than the non-commuting representative of a quantity does not change with time, this tells us that any information

we might have or obtain about this quantity (whether precise or of probability) will not change with time. (But patently this general information concerning constancy is about the nature, not the state.)

I must apologize for going into these details perhaps more than necessary, thereby deterring non-mathematical readers. I return to the main argument. A.m. has descended from celestial mechanics, initiated by NEWTON. The marvellous precision with which the motions of the heavenly bodies are predicted from Newton's laws — a precision unparalleled in any other branch of knowledge up to the present day — has made mechanics the prototype of exact physical science. Newton's pattern was closely followed in all the attempts of constructing models of the material world in order to account for its behaviour. It was followed not only as long as the hope or tendency prevailed to explain everything mechanically, but far beyond. For it does not really matter in principle (though the mathematical methods vary considerably), whether I give myself the initial positions and velocities of a number of particles that attract or repel each other by forces, known or assumed known, and ask myself what aspect will they offer at a given later time, — or whether my system includes field variables, distributed continuously throughout space and governed by laws that relate them to each other and to the motion of the particles. The close proximity to the Newtonian pattern consists in the peremptory demand that the said laws should, from a given initial state of particles and field, entail a definite state of the same at any later time, a definite orbit, as it were, of the whole system (notwithstanding the utter impossibility of actually checking the infinitely many data implied by even one such state).

5. – But Not of Physical Experimenting.

So this ideal of exactitude in physical science was inherited from astronomy: for any theory we think out the touchstone shall be that it enables us to predict the observable features of a physical system at any later time from sufficiently accurate observations made on the same system at an earlier time. This seems to be a sound basis for *thinking* about physical events, and I dare say the only sound one that has been conceived till now. If in our time it has been found out that nature is not such as to make accurate prediction possible in all cases, but sometimes only of probability, this is decidely of very great interest, but it does not change the pattern of thought fundamentally, provided the probabilities are predicted with accuracy (as is universally agreed that they are). At any rate this is *not* the point I wish to analyse here.

But the great difference between (positional) astronomy and physical science in general is this. Is astronomy, both before and after its fundamental law

had been discovered by NEWTON, the actual observations were and are of precisely the type of the ideal pattern (which, as I said, has been modelled after them). Several positions of a planet are observed — not just one, since its velocity is required, moreover only two angles are observable, while the third space coordinate must be inferred somehow. From these data later positions are computed and compared with observation. In this we assume Newton's law to be known. But even before it was known, the actual observations were of exactly the same kind. Only no very reliable prediction was possible. But it is known how KEPLER's genius succeeded in determining from a vast number of positional observations the actual facts, known as Keepler's laws, from which NEWTON read off, as it were, both the general law of motion and that of gravity, making thenceforward accurate prediction possible.

In physical science, however, as it has developed since, while this same pattern of *thought* has been copied and retained, it is found much too narrow, indeed mostly quite inadequate, to cover the actual observations. They are of entirely different, indeed of extremely multifarious types (as explained before in Section 3). Not only are we usually not in the post-Newtonian position of knowing the laws and testing them by prediction, but in the position of KEPLER. Our quest is after the *nature* of the system not after its *state*. Moreover for finding out what we want to know we do not follow the method of Kepler. We are not, or hardly ever, faced with a system that moves or changes its state of its own in a way that we would find out by carefully registering its observable features as functions of the time, as the positional astronomer does. I once had the good luck of having to supervise for three years an advanced practical (measuring) course in physics. Except for Atwood's machine (which was rather on an elementary side-track) and, perhaps, observations on a pendulum and the like, I do not remember a single experiment that followed these lines, but many, many along different lines. Now, this was in the early teens of this century; but I do not think the situation has changed since, neither in the courses of practical exercises, nor in the research laboratories.

6. – The Blind Spot in Quantum Mechanics.

Quantum mechanics (q.m.) has been shaped after analytical mechanics (a.m.), which in turn has descended from astronomy. Right at the outset the fascinating and intriguing novel feature presented itself, that the predictions of q.m. must not be regarded as unique but only as of probability. So much keen interest and honest work was spent on elaborating a scheme which fitted the new situation and yet remained close enough to its prototype (a.m.) for availing itself of its benefits, that no time or strength or inclin-

ation was left for noticing how far the methods of experimental laboratory research had drawn away from those of astronomy — to which they were never very close. Or was it believed that the new scheme (q.m.), that is a.m. readjusted so as to make only probability forecasts, was now equipped to apply *directly* to actual laboratory measurements (which a.m. never claimed except for simple cases as Atwood's machine or a pendulum)?

Anyhow, the claim *is* made. The new science (q.m.) arrogates the right to bully our whole philosophical outlook. It is pretended that refined measurements which lend themselves to easy discussion by the quantum-mechanical formalism could actually be made. They could not. (I am alluding to the gamma-ray-microscope, to the location of the electron in a « given » hydrogen atom, and the sort). Actual measurements on single individual systems are never discussed in this fundamental way, because the theory is not fit for it. This in itself is no blame. What is objectionable is the philosophical presumption, which claims reality for anything the quantum theorist chooses to imagine as measurable, while he closes his eyes to the fact that few, if any, actual measuring devices are amenable to discussion under his scheme.

One can certainly make a case for the view that the sum total of all observations which have been and ever will be made is after all the only reality, the only thing that physical science is concerned with. This view is not self-evident, but it is worth discussing. However to maintain the same about all observations that some school of theoreticians fancies, while in actual fact such observations are not made and differ in bulk from those that have been made and on which physical science is based, such a view is not founded on reason and cannot pretend to passing for serious philosophy. In using such plain language I hate to give offence to those of my friends who adhere to this kind of view (without realising that it is of this kind). But I wish to make it clear, that I shoulder now and ever after the full responsibility for my refractoriness. I am moving against the stream. But the tide will change.

7. – Our Objective is the General Laws.

At the end of Section 3 I promised to express in q.m.'s own language why most actual laboratory devices do not fit into its scheme of prophecy. When their right place within the accepted theory is pointed out, it becomes perfectly clear that and why they do not fit into the wrong place.

The situation is fairly obvious. The prophecy scheme (in all but a few outstanding exceptional instances, see Section 1) deals with measurements on systems whose *nature* is known. Experimental research is nearly always concerned with *finding out* the nature of the system under examination. It has its place earlier, by a well marked step, than the prophecy scheme. Its task

ranges with that of KEPLER, not with that of the astronomers after NEWTON. (Let it be mentioned, by the way, that they too have questions about the *nature* of their system left to them: the masses of the planets; the inertial frame; the appropriate time variable, since the rotation of the earth is not uniform.) To put it briefly: experimental research is interested in general laws, not in accidental *states*.

So is astronomy. But here it happens that the accidental state of the planetary system is of paramount practical importance for geography and for navigation. And, secondly, it so happens that a painstaking record of the time sequence of *states* is the only appropriate means for answering questions of *nature*, whether pre-Newtonian, as in the work of KEPLER, or post-Newtonian, as e.g. in ascertaining the tidal retardation of the rotating earth. The reason for this being so is that the astronomer has no means of interfering with his system: he can do nothing but observe it.

It might, of course, be the case, that in experimental physics the method for establishing general laws were the same as in astronomy. If this were so, the quantum mechanical theory of measurement might be all right. But it is not so. And that is small wonder. The physicist has full liberty to interfere with his object and to set the conditions of experiment at will. This empowers him to invent methods widely different from, and largely superior to, the placid observation of the astronomer. It is not astonishing that the strictly astronomical scheme of quantum mechanical prophecy is too narrow to embrace them.

In quantum mechanical language I would say, that the physicist's experiment is usually not aimed at finding out the state-function of his physical object, but at discovering characteristic features of its Hamiltonian (very often: its eigenvalues). For the Hamiltonian is the representative of the nature of the system, of the general laws that govern it in any state. Now I must repeat myself. It is perfectly thinkable that a good way of finding out about the Hamiltonian were the inversion of the prophecy scheme: you measure initial and final values many times and ask what Hamiltonian will correlate them correctly. If this were so (as it is in astronomy), the quantum mechanical theory of measurement might be all right. But it is not so. The fact that from a known Hamiltonian the prediction is only of probability makes the inverse problem exceedingly involved, as everybody who has an insight into the mathematics of the subject will admit. It is small wonder that the experimenter hardly ever follows this course. The most interesting questions are those about the discrete eigenvalues of some physical variable (mostly: the energy) or about some other matrix-elements of some such quantity (mostly: perturbation energy). These questions are sometimes answered by producing suitable experimental conditions repeatedly, never by following an individual system through a long course of its orbit, because this

is not possible. Repeated short period observations on similar systems are then put together and taken to form virtually the potential history of one and the same system.

The latter remark refers mainly to the tracing of the orbits of individual particles and of the events produced by them (as nuclear disintegrations) in the cloud chamber and in photographic emulsions. In these experiments we are in a similar position to that of the astronomer with regard to our not being able to influence the event. Yet the situation is not quite as bad, for we prescribe the medium in which the events take place (the nature and pressure of the gas or the composition of the photographic emulsion), and we can apply a magnetic field of known strength, which gives valuable information by curving the paths.

8. – Conclusion.

There is a habit in some quarters to answer objections of the kind raised here by saying that they are a matter of philosophical taste and not relevant to any question physics is really concerned with. This attitude is an instance of the fact that scientists are inclined to take their own outlook for the natural way of looking at things, while the outlooks of others, in asmuch as they differ from theirs, are adulterated by preconceived and unwarranted philosophical tenets, which unprejudiced science must avoid.

The ingegnious new-comer to quantum mechanics asks many inconvenient questions from which, in the considered opinion of the adepts, he must be weaned. He asks for instance whether the state-transitions in the atom that accompany the emission of a light-quant are instantaneous or whether they take time and pass through intermediate states. He is told that this question is meaningless and cannot be answered. Meaning is only attached to the value we find for the energy if we measure it, this can (by axiom) only be either the value of the initial state or that of the final state, the probability of finding the latter rather than the former increases with time continuously in a way that the theory foretells.

Another example: our bright disciple may find out for himself, that according to his theoretical instructions nothing prevents the velocity of a particle being measured by the time-honoured method which is practiced on the race-course and by the police (to trace offenders against the speed-limits), viz. by recording the time taken by the particle to cover a known distance; and he is perturbed in noticing that nothing is in the way of carrying the accuracy of this measurement far beyond the limit imposed by the Uncertainty Principle. The answer he gets from the initiates is, that this is indeed so,

but causes no worry, since the conflicting data refer to a bygone moment and cannot be used for predicting the future.

These examples could be multiplied. The answers are intriguing; they appear to be unassailable, for they seem to rest on the simple and safe principle that sound and sober reality, for the purposes of science, coincides with what is (or might be) observed. But actually this is not the whole story. We are also supposed to admit that the extent of what is, or might be, observed coincides exactly with what quantum mechanics is pleased to call observable. I have endeavoured to adumbrate here that it does not. And my point is that this is not an irrelevant issue of philosophical taste; it will compel us to recast the conceptual schema of quantum mechanics.

RIASSUNTO

L'interpretazione più comune della meccanica quantistica (m.q.) è interamente fondata sulla sua teoria della misura. Come sola realtà accessibile si considerano i risultati quantitativi delle osservazioni, il nostro unico fine essendo quello di predirli per quanto possibile a partire da altre osservazioni già fatte sullo stesso sistema fisico. Questo schema è interamente suggerito dalla astronomia di posizione, sul cui grande strumento analitico (la meccanica analitica) è stata modellata la stessa m.q.. Ma le esperienze di laboratorio ben di rado seguono lo schema dell'astronomia. L'astronomo non può che osservare i suoi oggetti, mentre il fisico può influenzare i propri in molte maniere, e anzi lo fa in modo elaborato. In astronomia la sequenza temporale degli *stati* è non solo di enorme interesse pratico (per esempio per la navigazione), ma è stata ed è il solo modo di scoprire la *legge*, che si è finito per conoscere nei suoi aspetti generali (Newton). Il fisico ancora oggi si propone di scoprire la legge (in terminologia tecnica: una Hamiltoniana); ma raramente, o mai, egli cerca di raggiungere lo scopo seguendo la successione temporale degli stati di un singolo sistema, che non sono di per sè di interesse fisico. L'interpretazione più comune della m.q. si vanta di essere intimamente legata alla scienza sperimentale. Ma in realtà è basata su uno schema di misura che, essendo interamente antiquato, è ben poco adatto a descrivere qualunque esperienza che venga realmente eseguita, ma piuttosto una schiera di esperienze per sempre limitate alla immaginazione dei loro inventori.

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The Stress Tensor in Field Dynamics.

P. A. M. Dirac

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Summary. -- The stress tensor is not completely determined by the usual conditions required of it. However, if one has a Hamiltonian formulation of the field equations on general space-like surfaces in space-time, one can connect the density of the Hamiltonian function, provided it satisfies a certain condition of curvature independence, with the stress tensor and so fix the latter completely. Conversely, if one is given a Hamiltonian formulation only on flat surfaces and one is also given the stress tensor, one can use it to provide the Hamiltonian formulation on general surfaces, provided the Poisson bracket relations of the stress tensor satisfy certain conditions equivalent to curvature independence.

1. – Introduction.

For a self-contained dynamical system the total energy and momentum are well-defined concepts and form, in relativistic theory, a four-vector P_μ ($\mu = 0, 1, 2, 3$). In a field dynamics one may localize the energy and momentum by introducing an energy-momentum density, $T_{\mu 0}$ say, at each point at a certain time, so that

(i) $$P_\mu = \int T_{\mu 0}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3 .$$

The laws of conservation of energy and momentum require that $\mathrm{d}P_\mu/\mathrm{d}x_0 = 0$, which shows that $\partial T_{\mu 0}/\partial x_0$ must equal a three-dimensional divergence, say $-\partial T_{\mu r}/\partial x_r$ ($r = 1, 2, 3$). Thus

(ii) $$\partial T_{\mu\nu}/\partial x_\nu = 0 .$$

The $T_{\mu r}$ introduced in this way form, together with $T_{\mu 0}$, a second rank tensor $T_{\mu\nu}$, the *stress tensor*.

The total angular momentum is another well-defined concept. In relativistic theory it forms three of the components of a six-vector $M_{\mu\nu} = -M_{\mu\nu}$, whose other three components are connected with Lorentz transformations. This six-vector is given in terms of the energy-momentum density $T_{\mu 0}$ by

$$M_{\mu\nu} = \int (x_\mu T_{\nu 0} - x_\nu T_{\mu 0})\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3 , \tag{iii}$$

an equation which shows how $M_{\mu\nu}$ is built up from localized contributions.

The law of conservation of angular momentum forms part of the law of conservation of $M_{\mu\nu}$, which requires $\mathrm{d}M_{\mu\nu}/\mathrm{d}x_0 = 0$. This leads, with the help of (iii) and (ii), to

$$T_{\mu\nu} = T_{\nu\mu} . \tag{iv}$$

The four conditions (i)–(iv) are the fundamental conditions that the stress tensor has to satisfy. They are not sufficient to determine it completely. Condition (i) leaves $T_{\mu 0}$ undetermined to the extent of an arbitrary three-dimensional divergence and the other conditions do not completely eliminate this arbitrariness, as one can see by noting that one can add to $T_{\mu\nu}$

$$\delta T_{\mu\nu} = \partial^2 S/\partial x^\mu\, \partial x^\nu - g_{\mu\nu}\, \partial^2 S/\partial x_\sigma\, \partial x^\sigma$$

for any scalar quantity S, without affecting the validity of any of the four conditions.

Hence in order to fix $T_{\mu\nu}$ some further condition is needed. Usually one has an action principle to give the field equations, and one can then define $T_{\mu\nu}$ in a simple way in terms of the action density $\mathcal{L}$ and its partial derivatives with respect to field quantities so as to satisfy the four conditions. (See F. J. Belinfante: *Physica*, **6**, 887 (1939)). With such a method the definition of $T_{\mu\nu}$ gets completed by the addition of a simplicity requirement.

Rosenfeld has improved on this procedure (L. Rosenfeld: *Mém. Acad. Roy. Belg.*, **18**, fasc. 6 (1940)). He works with a general gravitational field and defines $T_{\mu\nu} = \partial\mathcal{L}/\partial g^{\mu\nu}$ in a suitable variation process, which provides a definite stress tensor, independent of any simplicity argument. However, in atomic physics we are not interested in a gravitational field, so Rosenfeld's method then brings in an unnecessary complication and is not really suitable.

In atomic physics the Hamiltonian and the Poisson bracket relationships of the field quantities are all-important concepts. The present paper will be concerned with linking the stress tensor with these concepts and so completing its definition.

The Hamiltonian treatment of the equations of motion deals with the state at a certain time and discusses how the state changes when the time

is changed. In order to make the treatment relativistic, one must deal with states at a particular time relative to any observer. Such a state is to be pictured as the physical conditions on a flat three-dimensional « surface » in four-dimensional space-time. Any flat surface whose normal lies within the light-cone is a possible one, and the Hamiltonian equations of motion must tell us how the physical conditions change for any slight motion of the surface, which may be either a motion parallel to itself or a rotation.

The restriction to flat surfaces now appears undesirable, as it prevents one from making a slight change in the direction of the normal to the surface without giving a great motion to the surface at great distances. It leads to a theory which is unphysical, because the effects of conditions at great distances are too important.

One is therefore led to extend the Hamiltonian formulation so as to deal with states on general curved three-dimensional space-like surfaces, (the space-like condition meaning that the normal is always within the light-cone). Such a general surface is defined by giving the four coordinates y_μ of any point on it as functions of three parameter u_r. The $y_\mu(u)$ may be any functions of the u's, subject to the condition that there is a unit normal $l_\mu(u)$ at each point u, lying within the light-cone.

The conditions (i) and (iii) refer to the flat surface $x_0 =$ constant, and must now be generalized to refer to a general curved surface. They then read

(ia) $$P_\mu = \int T_{\mu\nu} l^\nu \Gamma \, \mathrm{d}^3 u \,,$$

(iiia) $$M_{\mu\nu} = \int (y_\mu T_{\nu\sigma} - y_\nu T_{\mu\sigma}) \, l^\sigma \Gamma \, \mathrm{d}^3 u \,,$$

where Γ is such that $\Gamma \mathrm{d}^3 u$ is an element of volume. In this form they are relativistic. $T_{\mu\nu} l^\nu \Gamma$ appears here as the energy-momentum density in the surface, with the word « density » referring to unit variation of the parameters u. The energy density in the surface is now $T_{\mu\nu} l^\mu l^\nu \Gamma$.

In a Hamiltonian theory dealing with general surfaces one requires a Hamiltonian function to generate the change in the physical conditions arising with any small deformation of the surface. If the deformation is a local one, the Hamiltonian function must be a local one, so there exists a *density of Hamiltonian function*. From the general connection between the Hamiltonian function and the energy, one would be inclined to assume that the density of Hamiltonian function equals the energy density $T_{\mu\nu} l^\mu l^\nu \Gamma$.

The main purpose of the present paper is to examine this assumption. It will be found to be justifiable provided the density of Hamiltonian function satisfies a certain condition of being independent of the curvature of the surface and its parametrization. The Hamiltonian formalism then fixes one

of the components of $T_{\mu\nu}$, and in a relativistic theory this suffices to fix them all.

2. – Displacements and Rotations.

For dealing with the general space-like surfaces we shall use the methods given in an earlier paper (P. A. M. DIRAC: *Can. Journ. of Maths.*, **3**, 1 (1951)), with some slight changes of notation.

The physical conditions on the surface will be described by certain basic dynamical variables, functions of the parameters u. Let us call them the p-variables. There will be definite P. b. relations between them in any Hamiltonian theory.

The y_μ which fix the surface are treated as further dynamical variables. They have zero P. b.'s with the p-variables and with one another. One must introduce dynamical variables w_μ as conjugates to the y_μ, satisfying the conditions that they have zero P. b.'s with the p-variables and with one another and

$$[y_\mu, w'_\nu] = \delta . \tag{1}$$

The δ here is short for the three-dimensional δ-function $\delta(u - u')$.

The complete set of basic dynamical variables now consists of the p-variables, the y's and the w's. Any dynamical variable occurring in the theory must be a function of these basic variables and their derivatives with respect to the u's.

A general motion of the surface and its parametrization is obtained by giving arbitrary values to $\mathrm{d}y_\mu/\mathrm{d}\tau = \dot{y}_\mu$. The Hamiltonian equations of motion describing such a change are, for any dynamical variable ξ,

$$\mathrm{d}\xi/\mathrm{d}\tau = [\xi, H] , \tag{2}$$

where

$$H = \int \dot{y}'_\nu \mathcal{H}'_\nu \, \mathrm{d}^3u' , \tag{3}$$

with $\mathcal{H}_\nu$ a given function of the dynamical variables, weakly equal to zero

$$\mathcal{H}_\nu \approx 0 . \tag{4}$$

The condition of being weakly equal to zero means that the equation can be used only after all P. b.'s have been worked out. Equations (2), (3) and (4) give

$$\mathrm{d}\xi/\mathrm{d}\tau \approx \int \dot{y}'_\nu [\xi, \mathcal{H}'_\nu] \, \mathrm{d}^3u' . \tag{5}$$

Putting $\xi = \mathcal{H}_\mu$, we get the consistency condition

$$[\mathcal{H}_\mu, \mathcal{H}'_\nu] \approx 0 . \tag{6}$$

Putting $\xi = y_\mu$, we see that

$$[y_\mu, \mathcal{H}'_\nu] = \delta . \tag{7}$$

Comparing this with (1), we get

$$\mathcal{H}_\nu = w_\nu + \beta_\nu , \tag{8}$$

where β_ν has zero P. b. with y_μ and is thus a function only of the y's and the p-variables and their derivatives with respect to the u's.

We now see that (6) can be strengthened to

$$[\mathcal{H}_\mu, \mathcal{H}'_\nu] = 0 , \tag{9}$$

since from (8) the left-hand side must be independent of the w's, and the only things in the theory that are weakly equal to zero and not strongly zero are linear functions of the $\mathcal{H}$'s, which must involve the w's.

Let us displace the whole surface by a small vector a_μ in space-time, keeping the parametrization and the physical conditions on the surface unchanged. This results in changing the variables y_μ by the constants a_μ and leaving the p-variables unchanged. Let us assume the w variables are also unchanged. Then the changes in all the dynamical variables are described by the infinitesimal contact transformation

$$\xi^* = \xi + \left[\xi, a_\mu \int w'_\mu \, \mathrm{d}^3 u'\right] .$$

The generator of this transformation, namely

$$a_\mu \int w'_\mu \, \mathrm{d}^3 u' , \tag{10}$$

is the *surface displacement operator* for the displacement a_μ.

The operator

$$a_\mu \int \mathcal{H}'_\mu \, \mathrm{d}^3 u' \tag{11}$$

generates an infinitesimal contact transformation in which the whole surface is displaced by a_μ and the p-variables get changed in accordance with the equations of motion to correspond to a fixed physical state of motion. Thus the difference (11) — (10) or

$$a_\mu \int \beta'_\mu \, \mathrm{d}^3 u' \tag{12}$$

generates an infinitesimal contact transformation in which the surface is unchanged and the p-variables get changed so as to make them correspond to a physical state of motion displaced by $-a_\mu$. From the general dynamical theory connecting displacement with energy and momentum, (12) must equal $a_\mu P_\mu$, so

$$P_\mu = \int \beta'_\mu \, \mathrm{d}^3u' . \tag{13}$$

One can understand these connections better by comparing them with their analogues in non-relativistic particle dynamics. Here we have a Hamiltonian function $H(q, p)$ generating the equations of motion for the dynamical variables q, p. We may count the time t as an extra dynamical coordinate and introduce a momentum w conjugate to it, and then the operator $F=H+w$ generates the equations of motion for all the variables. Now F, which is weakly equal to zero, is the analogue of $\int \mathcal{H}'_0 \, \mathrm{d}^3u'$, w is the analogue of $\int w'_0 \, \mathrm{d}^3u'$ and H is the analogue of $\int \beta'_0 \, \mathrm{d}^3u'$.

There are corresponding relationships connecting rotations with the angular momentum. For discussing rotations it is convenient to take p-variables that do not refer to the directions of the x_μ-axes: if they refer to any directions at all, these must be the directions of the normal to the surface and of the u-axes in the surface. One can always choose such p-variables. Then any dynamical variable independent of the w's that involves a suffix μ can do so only by containing y_μ or some derivative of y_μ with respect to the u's.

If we now apply to the whole surface a small rotation in four dimensions about the origin $x_\mu = 0$ and keep the physical conditions in the surface unaltered relative to the surface, the change in all the dynamical variables is given by the infinitesimal contact transformation

$$\xi^* = \xi + \left[\xi, \tfrac{1}{2} m_{\mu\nu} \int (y'_\mu w'_\nu - y'_\nu w'_\mu) \, \mathrm{d}^3u'\right] ,$$

where $m_{\mu\nu} = -m_{\nu\mu}$ is a small six-vector which specifies the rotation. The operator

$$\tfrac{1}{2} m_{\mu\nu} \int (y'_\mu w'_\nu - y'_\nu w'_\nu) \, \mathrm{d}^3u' \tag{14}$$

which generates the transformation is the *surface rotation operator.*

The operator

$$\tfrac{1}{2} m_{\mu\nu} \int (y'_\mu \mathcal{H}'_\nu - y'_\nu \mathcal{H}'_\mu) \, \mathrm{d}^3u' \tag{15}$$

generates the same rotation of the surface, combined with the change in the

p-variables needed to keep to a fixed physical state of motion. The difference (15) — (14) or

$$\frac{1}{2}m_{\mu\nu}\int(y'_\mu\beta'_\nu - y'_\nu\beta'_\mu)\,\mathrm{d}^3u' \tag{16}$$

generates no change of the surface, but gives the change of the physical conditions in the surface corresponding to the application of the small rotation $-m_{\mu\nu}$ to the state of motion. From the general connection between rotations and angular momentum, (16) must equal $\frac{1}{2}m_{\mu\nu}M_{\mu\nu}$. Thus

$$M_{\mu\nu} = \int(y'_\mu\beta'_\nu - y'_\nu\beta'_\mu)\,\mathrm{d}^3u'. \tag{17}$$

3. – Invariance Properties of Dynamical Variables.

We shall consider certain invariance properties that may be possessed by a dynamical variable ξ located at a particular point u in the surface.

It may be that ξ is unaffected by any change of the parametrization of the surface in the neighbourhood of the point u that does not involve the point u itself being shifted. In this case ξ is called a *u-scalar*. The mathematical condition has been worked out (*Can. Journ. of. Maths.*, **3**, 10, where φ is used instead of the present $\mathcal{H}$), and is

$$[\xi, \mathcal{H}'^r] = \xi^r\delta\,. \tag{18}$$

If ξ divided by Γ is a u-scalar, ξ is called a u-scalar density. The mathematical condition is

$$[\xi, \mathcal{H}'^r] = \xi'\delta^r = (\xi\delta)^r\,. \tag{19}$$

The dynamical variable ξ may be *surface independent*, which means that it is unaltered by any deformation of the surface such that the point u is kept fixed. The condition to be surface independent is stronger than the condition to be a u-scalar, because it is necessary that ξ should also be unaffected by a change in the direction of the normal to the surface. The condition for ξ to be surface independent is the necessary and sufficient condition for it to be a quantity with a physical meaning located at a point in space-time, i.e. a field quantity.

The mathematical condition for ξ to be surface independent is that $\mathrm{d}\xi/\mathrm{d}\tau$ shall vanish for any motion of the surface such that $\dot{y}_\nu = 0$ at the particular point u at which ξ is located. From (5), the condition becomes

$$[\xi, \mathcal{H}'_\nu] = a_\nu\delta \tag{20}$$

with some coefficient a_ν. The important feature of this equation is the absence of terms with derivatives of δ.

For a general motion of the surface, with $\dot{y}'_\nu$ arbitrary for all u', we must have

$$\mathrm{d}\xi/\mathrm{d}\tau = \dot{y}_\nu \xi_\nu ,$$

where ξ_ν is $\mathrm{d}\xi/\mathrm{d}x_\nu$ at the point u on the surface. Comparing this with the right-hand side of (5), we get

$$[\xi, \mathcal{H}'_\nu] = \xi_\nu \delta , \tag{21}$$

which gives us the meaning of the coefficient a_ν. It is evident physically, and can easily be checked mathematically by an application of Poisson's identity, that ξ_ν is also surface independent.

The mathematical condition (21) for surface independence can be divided into two conditions

$$[\xi, \mathcal{H}'^{r}] = y_\mu{}^r \xi_\mu \delta = \xi^r \delta ,$$

$$[\xi, \mathcal{H}'_l] = l_\mu \xi_\mu \delta = \xi_l \delta . \tag{22}$$

The first is the condition for ξ to be a u-scalar and the second is the condition for it to be independent of the direction of the normal to the surface.

A weaker property than surface independence is that of *curvature independence*. ξ is curvature independent if it is unaltered by any deformation of the surface which leaves unaltered y_μ and its first derivatives $y_\mu{}^r$ at the point u where ξ is located. The mathematical condition for this is found to be, by the same argument that led to (20),

$$[\xi, \mathcal{H}'_\nu] = a_\nu \delta + b_{\nu r} \delta^r , \tag{23}$$

with some coefficients a_ν, $b_{\nu r}$. The important feature here is the absence of terms with second or higher derivatives of δ.

4. – Passage from the Hamiltonian to the Stress Tensor.

Suppose we are given a Hamiltonian formulation of a field theory on general surfaces, so that we have given functions β_μ defined by (8). They form the density of Hamiltonian function which generates the change in the physical conditions arising with a local deformation of the surface and its parametrization. The normal component β_l corresponds to a motion of an element of the surface normal to itself, and the tangential component β^r to a local

change of parametrization. One would be inclined to assume that they are respectively equal to the energy density and momentum density in the surface, so that we have

$$\beta_\mu = T_{\mu\nu} l_\nu \Gamma . \tag{24}$$

This assumption would actually make (13) go over into (ia) and (17) into (iiia).

To examine the legitimacy of this assumption, we must see whether we can introduce $T_{\mu\nu}$ to satisfy (24) and at the same time to be surface independent. It will be shown that we can, provided we assume that β_μ is curvature independent.

Some of the conditions needed for β_μ to be curvature independent are necessarily fulfilled on account of the consistency conditions (9). These give

$$[\mathcal{H}_\mu, \mathcal{H}'^r] = [\mathcal{H}_\mu, y'_\nu{}^r] \mathcal{H}'_\nu = \mathcal{H}'_\mu \delta^r ,$$

showing that $\mathcal{H}_\mu$ is a u-scalar density. Since w_μ is also a u-scalar density, their difference β_μ is a u-scalar density, i.e.

$$[\beta_\mu, \mathcal{H}'^r] = \beta_\mu{}' \delta^r . \tag{25}$$

The only further condition that we need assume is thus that $[\beta_\mu, \mathcal{H}'_l]$ is of the form of the right-hand side of (23), say

$$[\beta_\mu, \mathcal{H}'_l] = q_\mu \delta + n'_{\mu r} \delta^r . \tag{26}$$

We suppose that we are dealing with a self-contained dynamical system so that $\mathcal{H}_l$ cannot involve the y_μ themselves, but only their derivatives with respect to the u's. Then

$$\left[\int w_\mu \, \mathrm{d}^3 u, \ \mathcal{H}'_l\right] = 0 ,$$

Now

$$\left[\int \mathcal{H}_\mu \, \mathrm{d}^3 u, \ \mathcal{H}'_l\right] \approx 0 ,$$

and hence

$$\left[\int \beta_\mu \, \mathrm{d}^3 u, \ \mathcal{H}'_l\right] = 0 . \tag{27}$$

It follows that q_μ must vanish, so that (26) reduces to

$$[\beta_\mu, \mathcal{H}'_l] = n'_{\mu r} \delta^r , \tag{28}$$

We shall need to work out some consequences of equations (25) and (28) in which the suffix μ plays no part. It is therefore convenient to drop this suffix for a while and rewrite these equations as

$$[\beta, \mathcal{H}^{\prime\prime}] = \beta^{\prime}\delta^{r}, \tag{29}$$

$$[\beta, \mathcal{H}_{l}^{\prime}] = n_{r}^{\prime}\delta^{r}. \tag{30}$$

Put

$$[n_r, \mathcal{H}_{l}^{\prime}] = a_r\delta + b_{rs}\delta^{s} + c_{rst}\delta^{st} + \cdots. \tag{31}$$

Then

$$\begin{aligned}[[\beta, \mathcal{H}_{l}^{\prime}], \mathcal{H}_{l}^{\prime\prime}] &= [n_{l}^{\prime}, \mathcal{H}_{l}^{\prime\prime}]\delta^{r}(u - u^{\prime}) \\ &= a_{r}^{\prime}\Delta - b_{rs}^{\prime}\Delta^{rs^{\prime\prime}} + c_{rst}^{\prime}\Delta^{rs^{\prime\prime}t^{\prime\prime}} - \cdots, \end{aligned} \tag{32}$$

where

$$\Delta = \delta(u - u^{\prime})\,\delta(u^{\prime} - u^{\prime\prime}) \tag{33}$$

and the suffixes attached to it denote derivatives with respect to the corresponding u variables. Note that Δ is symmetrical between u, $u^{\prime}$, and $u^{\prime\prime}$. Interchanging $u^{\prime}$ and $u^{\prime\prime}$ in (32), we get

$$[[\beta, \mathcal{H}_{l}^{\prime\prime}], \mathcal{H}_{l}^{\prime}] = a_{r}^{\prime\prime}\delta - b_{rs}^{\prime\prime}\Delta^{rs^{\prime}} + c_{rst}^{\prime\prime}\Delta^{rs^{\prime}t^{\prime}} - \cdots. \tag{34}$$

Further

$$\begin{aligned}[\beta, [\mathcal{H}_{l}^{\prime}, \mathcal{H}_{l}^{\prime\prime}]] &= -[\beta, \mathcal{H}_{s}^{\prime} + \mathcal{H}_{s}^{\prime\prime}]\delta^{s^{\prime}}(u^{\prime} - u^{\prime\prime}) \\ &= -\{\gamma_{rs}^{\prime}\beta^{\prime}\delta^{r}(u - u^{\prime}) + \gamma_{rs}^{\prime\prime}\beta^{\prime\prime}\delta^{r}(u - u^{\prime\prime})\}\delta^{s^{\prime}}(u^{\prime} - u^{\prime\prime}) \\ &= \gamma_{rs}^{\prime}\beta^{\prime}\Delta^{rs^{\prime\prime}} - \gamma_{rs}^{\prime\prime}\beta^{\prime\prime}\Delta^{rs^{\prime}}. \end{aligned} \tag{35}$$

If we now form the equation (34)+(35) − (32), the left-hand side vanishes from Poisson's identity and we are left with

$$0 = (\gamma_{rs}^{\prime}\beta^{\prime} + b_{rs}^{\prime})\Delta^{rs^{\prime\prime}} - (\gamma_{rs}^{\prime\prime}\beta^{\prime\prime} + b_{rs}^{\prime\prime})\Delta^{rs^{\prime}} - c_{rst}^{\prime}\Delta^{rs^{\prime\prime}t^{\prime\prime}} + c_{rst}^{\prime\prime}\Delta^{rs^{\prime}t^{\prime}} - \cdots. \tag{36}$$

The derivatives of Δ are connected by the identity

$$\Delta^{r} + \Delta^{r^{\prime}} + \Delta^{r^{\prime\prime}} = 0 \tag{37}$$

and identities obtained by differentiating this one, but are otherwise independent. These identities do not provide any connection between those deri-

vatives of Δ that occur in (36), except to show that $\Delta^{rs''}+\Delta^{rs'}$ equals $-\Delta^{rs}$ and is thus symmetrical between r and s. We can infer that all the coefficients in (36) vanish separately. Thus c and the later coefficients in the expansion (31) must vanish, and

$$b_{rs} = -\gamma_{rs}\beta\,, \tag{38}$$

so (31) becomes

$$[n_r, \mathcal{H}'_l] = a_r\delta - \gamma_{rs}\beta\delta^s\,. \tag{39}$$

A further use of Poisson's identity enables us to see that $n_r y_{r'}$ is a u-scalar density. We have

$$[[\beta, \mathcal{H}'_l]\mathcal{H}''^s] = [\beta, [\mathcal{H}'_l, \mathcal{H}''^s]] + [[\beta, \mathcal{H}''^s], \mathcal{H}'_l]\,.$$

which gives

$$\begin{aligned}
[n'_r, \mathcal{H}''^s]\delta^r(u-u') &= [\beta, \mathcal{H}''_l]\delta^{s'}(u'-u'') + [\beta'', \mathcal{H}'_l]\delta^s(u-u'') \\
&= n''_r\,\delta^r(u-u'')\,\delta^{s'}(u'-u'') + n'_r\delta^{r''}(u'-u'')\,\delta^s(u-u'') \\
&= n_r{}''\,\Delta^{rs'} - n'_r\Delta^{r's} \\
&= -n''_r(\Delta^{rs}+\Delta^{rs''}) + n'_r(\Delta^{rs}+\Delta^{r''s}) \\
&= n''_r\,\delta^r(u-u')\,\delta^{s'}(u'-u'') - n'_r\delta^s(u-u')\,\delta^{r'}(u'-u'')\,.
\end{aligned}$$

Thus

$$[n'_r, \mathcal{H}''^s] = n''_r\,\delta^{s'}(u'-u'') - n'_p\gamma^s_r\delta^{p'}(u'-u'')\,.$$

Hence

$$\begin{aligned}
[n_r y_{v'}{}^r, \mathcal{H}'^s] &= y_{v'}{}^r(n'_r, \delta^s - n_p\gamma^s_r\delta^p) + n_r(y_{v'}{}^s\delta)^r \\
&= y_{v'}{}^r(n_r\delta)^s + n_r y_{v'}{}^{sr}\delta \\
&= (n_r y_{v'}{}^r\delta)^s\,,
\end{aligned} \tag{40}$$

which was to be proved.

Define T_v by

$$\Gamma T_v = \beta l_v - n_r y_v{}^r\,. \tag{41}$$

From (40) T_v must be a u-scalar. Further, from (30) and (39),

$$\begin{aligned}
[\Gamma T_v, \mathcal{H}'_l] &= n'_r l_v\,\delta^r - \beta y_{vr}\delta^r - y_v{}^r(a_r\delta - \gamma_{rs}\beta\delta^s) - n_r(l_v\delta)^r \\
&= (n_r{}^r l_v - a_r y_v{}^r - n_r l_v{}^r)\delta\,,
\end{aligned} \tag{42}$$

which, combined with the formula

$$[\Gamma, \mathcal{H}'_l] = \Gamma \Omega^r{}_r \delta \tag{43}$$

leads to

$$\Gamma[T_\nu, \mathcal{H}'_l] = (n_r{}^r l_\nu - a_r y_\nu{}^r - n_r l_\nu{}^r - \Gamma T_\nu \Omega^r{}_r)\delta\,. \tag{44}$$

The absence of derivatives of δ here shows that T_ν satisfies the condition (22) and is therefore surface independent.

Let us now restore the suffix μ. Equation (41) becomes

$$\Gamma T_{\mu\nu} = \beta_\mu l_\nu - n_{\mu r} y_\nu{}^r \tag{45}$$

and constitutes, together with the equation (28) that defines $n_{\mu r}$, the definition of the stress tensor $T_{\mu\nu}$. The components $T_{\mu l} = T_{\mu\nu} l_\nu$ are given by (24) and the components $T_{\mu r} = T_{\mu\nu} y_{\nu r}$ are given by

$$[\beta_\mu, \mathcal{H}'_l] = -(\Gamma T_{\mu r}\delta)^r\,. \tag{46}$$

Let us analyse the condition for β_μ to be curvature independent, i.e. equation (26). We have

$$[w_l, w'_\mu] = w_\nu [l_\nu, w'_\mu] = -l_\mu w_r \delta^r\,,$$

and similarly

$$[\mathcal{H}_l, \mathcal{H}'_\mu] = -l_\mu \mathcal{H}_r \delta^r\,.$$

Subtracting, we get

$$[\mathcal{H}_l, \beta'_\mu] + [\beta_l, w'_\mu] = -l_\mu \beta_r \delta^r\,.$$

Thus the condition (26) leads to

$$[\beta_l, w'_\mu] = e\delta + f_r \delta^r\,, \tag{47}$$

with some coefficients e, f_r. This means that β_l, considered as a function of the y_μ and the p-variables and their derivatives with respect to the u's, must not contain any derivatives of y_μ higher than the first.

The result of the present section is to show that, when this condition is satisfied, one can introduce a surface independent stress tensor $T_{\mu\nu}$ by means of equation (45) or equations (24) and (46).

5. – The Conservation Laws.

The general statement of the conservation laws is that P_μ and $M_{\mu\nu}$, defined by (13) and (17), are the same for all surfaces. We shall examine the validity of these laws and see how they lead to the conditions (ii) and (iv) for the stress tensor.

In order that P_μ and $M_{\mu\nu}$ may be invariant for any deformation of the surface, they must have zero P. b. with $\mathcal{H}_\sigma$,

$$[\mathcal{H}_\sigma, P_\mu] = 0\,, \qquad [\mathcal{H}_\sigma, M_{\mu\nu}] = 0\,.$$

These equations can be resolved into tangential parts

$$[\mathcal{H}^r, P_\mu] = 0\,, \qquad [\mathcal{H}^r, M_{\mu\nu}] = 0\,, \tag{48}$$

and normal parts

$$[\mathcal{H}_l, P_u] = 0\,, \tag{49}$$

$$[\mathcal{H}_l, M_{\mu\nu}] = 0\,. \tag{50}$$

Equations (48) merely express that P_μ and $M_{\mu\nu}$ are invariant under change of parametrization of the surface and have no physical content. They are obviously satisfied on account of the integrands in (13) and (17) being u-scalar densities. Equation (49) follows directly from (28) and is a consequence of the dynamical system being self-contained, with no external fields. Equation (50) needs further consideration.

From (9) we get

$$[\mathcal{H}_l, M_{\mu\nu}] \approx -\left[\mathcal{H}_l, \int (y'_\mu w'_\nu - y'_\nu w'_\mu)\, \mathrm{d}^3u'\right].$$

It is easily verified that

$$\left[w_l, \int (y'_\mu w'_\nu - y_\nu' w'_\mu)\, \mathrm{d}^3u'\right] = 0\,, \tag{51}$$

and hence the condition (50) becomes

$$\left[\beta_l, \int (y'_\mu w'_\nu - y'_\nu w'_\mu)\, \mathrm{d}^3u'\right] = 0\,. \tag{52}$$

This means that β_l shall not be altered by a rotation applied to the y_μ variables.

We already have the condition that β_l can involve y_μ only through its first derivatives $y_\mu{}^r$. (52) imposes the further requirement that it shall involve these first derivatives only in combinations that do not contain a non-dummy suffix μ. Such combinations can only be $y_\mu{}^r y_\mu{}^s = \gamma^{rs}$.

In a relativistic theory β_l can involve a quantity $y_{\mu}{}^r$ only if it is multiplied into some factor also having the suffix μ, so as to make μ a dummy. We are assuming that none of the basic p-variables contains a suffix μ, so this factor can only be another of the quantities $y_\mu{}^s$. The condition (52) is then satisfied and the conservation of $M_{\mu\nu}$ is assured.

The situation would be different if some of the p-variables did contain a suffix μ, for example if some of them were the four potentials A_μ describing the electromagnetic field. In this case the argument leading to (17) would not be valid and the formula for $M_{\mu\nu}$ would need some extra terms. Some further work would then be needed to establish the conservation of this $M_{\mu\nu}$. The most convenient way of proceeding is now to make a contact transformation to new basic variables $A^r = A_\mu y_\mu{}^r$, $A_l = A_\mu l_\mu$, which would involve a change in the w variables, and then to use equation (17) and the above proof of the conservation of $M_{\mu\nu}$.

The conservation laws may be expressed as local conditions and are then (ii) and (iv.) To deduce (ii) we work from equation (44), from which the suffix μ has been dropped. It gives

$$\Gamma[T_\nu, l'_\nu \mathcal{H}'_l] = (n_r{}^r - \beta\Omega_r{}^r)\delta\,. \tag{53}$$

Since T_ν is a u-scalar,

$$\begin{aligned}\Gamma[T_\nu, y'_{\nu s'}\mathcal{H}'^s] &= \Gamma y_{\nu s} T_\nu{}^s \delta \\ &= \Gamma y_{\nu s}\left\{(\beta l_\nu - n_r y_\nu{}^r)/\Gamma\right\}^s \delta \\ &= \{\beta\Omega_s{}^s - n_r{}^s\gamma_s{}^r\}\,\delta\,,\end{aligned} \tag{54}$$

with the help of the formula

$$y_{\nu s}(y_\nu{}^r/\Gamma)^s = 0\,, \tag{55}$$

which follows from

$$\Gamma^r/\Gamma = \tfrac{1}{2}\gamma^{stv}\gamma_{st} = y_\nu{}^{sr} y_\nu{}^t \gamma_{st} = y_\nu{}^{rs} y_{\nu s}\,. \tag{56}$$

Adding (53) and (54) and restoring the suffix μ, we get

$$\Gamma[T_{\mu\nu}, l'_\nu \mathcal{H}'_l + y'_{\nu s'}\mathcal{H}'^s] = 0\,,$$

or

$$[T_{\mu\nu}, \mathcal{H}'_\nu] = 0\,.$$

Equation (ii) follows, with the help of (21).

To deduce (iv) we see that, from (28),

$$[y_\mu\beta_\nu - y_\nu\beta_\mu, \mathcal{H}'_l] = \beta_\nu l_\mu\delta + y_\mu(n_{\nu r}\delta)^r - \beta_\mu l_\nu\delta - y_\nu(n_{\mu r}\delta)^r$$
$$= \Gamma(T_{\nu\mu} - T_{\mu\nu})\delta + (y_\mu n_{\nu r}\delta)^r - (y_\nu n_{\mu r}\delta)^r .$$

Integrating with respect to the u's, we get

$$[M_{\mu\nu}, \mathcal{H}'_l] = \Gamma'(T'_{\nu\mu} - T'_{\mu\nu}) . \tag{57}$$

Thus the conservation law (50) leads directly to (iv).

This completes the proof that the stress tensor $T_{\mu\nu}$ introduced in the preceding section satisfies all the conditions required of it. It is determined by a curvature-independent Hamiltonian formulation of the equations of motion on general surfaces. It is not necessarily uniquely determined, because there may be more than one curvature-independent Hamiltonian formulation of the equations of motion, connected one with another by contact transformations. Each such Hamiltonian formulation will correspond to its own stress tensor.

6. – Passage ftom the Stress Tensor to the General Hamiltonian.

We have been considering the problem of starting with a Hamiltonian formulation on general curved surfaces and seeing how to introduce the stress tensor $T_{\mu\nu}$ in terms of the density of Hamiltonian function β_μ. We shall now consider a converse problem. Let us suppose we are given a Hamiltonian formulation on flat surfaces only and suppose we are also given a $T_{\mu\nu}$ satisfying the four conditions (i)–(iv). Can we then take $T_{\mu t}\Gamma$ as the density of Hamiltonian function β_μ for a Hamiltonian formulation on general curved surfaces?

The assumption of a Hamiltonian formulation on flat surfaces means we are given certain field quantities satisfying given field equations and with given P. b. relationships between them, and we are also given the total energy and momentum P_μ and the six-vector $M_{\mu\nu}$ such that, for any field quantity ξ,

$$[\xi, P_\mu] = \xi_\mu \tag{58}$$

and $\frac{1}{2}m_{\mu\nu}[\xi, M_{\mu\nu}]$ equals the change in ξ when one applies to it the small rotation $m_{\mu\nu}$. The given P. b. relationships must, of course, all be consistent, i.e. they must satisfy Poisson's identity.

We introduce a general curved surface and take as dynamical variables the y_μ that define the surface and the various field quantities on the surface. The y_μ are assumed to have zero P. b's with the field quantities and with

one another. We now introduce further dynamical variables $\mathcal{H}_\mu$ that are assumed to satisfy

$$[\mathcal{H}_\mu, \mathcal{H}'_\nu] = 0 , \tag{59}$$

and

$$[\xi, \mathcal{H}'_\nu] = \xi_\nu \delta , \tag{60}$$

where ξ is y_μ or any field variable. It is easily verified that these P. b. relations are consistent, provided the P. b. relations between the field quantities are consistent with the field equations. Equations (59), (60) define $\mathcal{H}_\mu$ only to the extent that they fix its P. b. with any quantity. We make the further assumption that $\mathcal{H}_\mu$ is weakly zero, and then it is defined as completely as any quantity is in Hamiltonian theory.

We assume we are given a $T_{\mu\nu}$ satisfying the four conditions (i)–(iv). The conditions (ia), (iiia) are deducible from them. Let us now define β_μ by

$$\beta_\mu = T_{\mu\nu} l_\nu \Gamma \tag{61}$$

and define w_μ by

$$w_\mu = \mathcal{H}_\mu - \beta_\mu . \tag{62}$$

It is then easily seen that we have all the conditions for a Hamiltonian formulation of the field equations on the general surface, with the exception of

$$[w_\mu, w'_\nu] = 0 . \tag{63}$$

It will now be proved that the condition (63) holds, provided the P. b. relations of the β's are of the form

$$[\beta_\mu, \beta'_\nu] = a_{\mu\nu}\delta + b_{\mu\nu\tau}\delta^\tau \tag{64}$$

with no derivatives of δ higher than the first. The assumption (64) plays the role of the curvature-independent assumption that was needed in § 4.

Equation (60) applied to the field variable $T_{\mu\sigma}$ gives

$$[T_{\mu\sigma}, \mathcal{H}'_\nu] = T_{\mu\sigma\nu}\delta . \tag{65}$$

Now $[l_\sigma, \mathcal{H}'_\nu]$ and $[\Gamma, \mathcal{H}'_\nu]$ do not contain derivatives of δ higher than the first, so from (61) $[\beta_\mu, \mathcal{H}'_\nu]$ does not contain derivatives of δ higher than the first. From (62)

$$[w_\mu, w'_\nu] = [\mathcal{H}_\mu, \mathcal{H}'_\nu] - [\beta_\mu, \mathcal{H}'_\nu] - [\mathcal{H}_\mu, \beta'_\nu] + [\beta_\mu, \beta'_\nu] .$$

Hence the assumption (64) leads to

$$[w_\mu, w'_\nu] = h_{\mu\nu}\delta + k_{\mu\nu r}\delta^r \tag{66}$$

with certain coefficients h and k.

The conservation of P_ν is known, because we are assuming (ii). Hence

$$\left[\mathcal{H}_\mu, \int \beta'_\nu \, \mathrm{d}^3u'\right] = 0 ,$$

so (59) leads to

$$\left[\mathcal{H}_\mu, \int w'_\nu \, \mathrm{d}^3u'\right] = 0 . \tag{67}$$

Integrating (65) with respect to the u''s and subtracting (58), with ν written for μ, applied to $\xi = T_{\mu\sigma}$, we get

$$\left[T_{\mu\sigma}, \int w'_\nu \, \mathrm{d}^3u'\right] = 0 .$$

Hence

$$\left[\beta_\mu, \int w'_\nu \, \mathrm{d}^3u'\right] = 0 . \tag{68}$$

Equations (67) and (68) give

$$\left[w_\mu, \int w'_\nu \, \mathrm{d}^3u'\right] = 0 .$$

Thus the coefficient h in (66) must vanish and we are left with

$$[w_\mu, w'_\nu] = k_{\mu\nu r}\delta^r . \tag{69}$$

If we interchange μ, ν and also u, u' in (69), we get

$$[w'_\nu, w_\mu] = -k'_{\nu\mu l}\delta^l = -(k_{\nu\mu r}\delta)^r .$$

In order that this may agree with (69), we must have

$$k_{\mu\nu r} = k_{\nu\mu r} , \tag{70}$$

and

$$k_{\mu\nu r}{}^r = 0 . \tag{71}$$

By considering the effect of a small rotation applied to $T_{\rho\sigma}$, we get

$$[T_{\rho\sigma}, M_{\mu\nu}] = y_\mu T_{\rho\sigma\nu} + g_{\rho\mu}T_{\nu\sigma} + g_{\sigma\mu}T_{\rho\nu} - ,$$

where the — at the end of the right-hand side means that we must subtract all the preceding terms with μ and ν interchanged. From (60) applied to $T_{\varrho\sigma}$,

$$\left[T_{\varrho\sigma}, \int (y'_\mu \mathcal{H}'_\nu - y'_\nu \mathcal{H}'_\mu)\, \mathrm{d}^3u'\right] = y_\mu T_{\varrho\sigma\nu} - .$$

Subtracting, we get

$$\left[T_{\varrho\sigma}, \int (y'_\mu w'_\nu - y'_\nu w'_\mu)\, \mathrm{d}^3u'\right] = - g_{\varrho\mu} T_{\nu\sigma} - g_{\sigma\mu} T_{\varrho\nu} - . \tag{72}$$

Now it is easily seen that

$$\left[\Gamma, \int (y'_\mu w'_\nu - y'_\nu w'_\mu)\, \mathrm{d}^3u'\right] = 0$$

and

$$\left[l_\sigma, \int (y'_\mu w'_\nu - y'_\nu w'_\mu)\, \mathrm{d}^3u'\right] = g_{\sigma\nu} l_\mu - g_{\sigma\mu} l_\nu ,$$

so (72) leads to

$$\left[\beta_\varrho, \int (y'_\mu w'_\nu - y'_\nu w'_\mu)\, \mathrm{d}^3u'\right] = - g_{\varrho\mu}\beta_\nu + g_{\varrho\nu}\beta_\mu . \tag{73}$$

We have

$$\begin{aligned} [\beta_\nu, \mathcal{H}'_\varrho] &= [\Gamma T_{\nu\sigma} l_\sigma, \mathcal{H}'_\varrho] \\ &= \Gamma T_{\nu\sigma\varrho} l_\sigma \delta + \Gamma T_{\nu\sigma}(l_\sigma y_{\varrho s} - l_\varrho y_{\sigma s})\, \delta^s . \end{aligned} \tag{74}$$

The formula (55) shows that $(\Gamma y_{\nu s})^s$ is a multiple of l_ν. It follows that

$$\begin{aligned} \{\Gamma T_{\nu\sigma}(l_\sigma y_{\varrho s} - l_\varrho y_{\sigma s})\}^s &= \Gamma(T_{\nu\sigma} l_\sigma)^s y_{\varrho s} - \Gamma(T_{\nu\sigma} l_\varrho)^s y_{\sigma s} \\ &= \Gamma T_{\nu\sigma}{}^s (l_\sigma y_{\varrho s} - l_\varrho y_{\sigma s}) \\ &= \Gamma(T_{\nu\sigma\varrho} l_\sigma - T_{\nu\sigma\sigma} l_\varrho) . \end{aligned} \tag{75}$$

Thus, with the help of (ii), (74) becomes

$$[\beta_\nu, \mathcal{H}'_\varrho] = \{\Gamma T_{\nu\sigma}(l_\sigma y_{\varrho s} - l_\varrho y_{\sigma s})\, \delta\}^s ,$$

so from (59)

$$[\mathcal{H}_\varrho, w'_\nu] = - \Gamma T_{\nu\sigma}(l_\sigma y_{\varrho s} - l_\varrho y_{\sigma s})\delta^s .$$

This gives

$$\left[\mathcal{H}_\varrho, \int y'_\mu w'_\nu \,\mathrm{d}^3u'\right] = -\int \{g_{\varrho\mu} w'_\nu \delta + y'_\mu \Gamma T_{\nu\sigma}(l_\sigma y_{\varrho s} - l_\varrho y_{\sigma s})\,\delta^s\}\,\mathrm{d}^3u'$$
$$= -g_{\varrho\mu} w_\nu - \Gamma T_{\nu\sigma}(l_\sigma y_{\varrho s} - l_\varrho y_{\sigma s}) y_\mu{}^s$$
$$= -g_{\varrho\mu} w_\nu - g_{\varrho\mu}\beta_\nu + \Gamma T_{\nu\mu} l_\varrho \,.$$

By subtracting from this equation the result of interchanging μ and ν in it and then subtracting (73), we find

$$\left[w_\varrho, \int (y'_\mu w'_\nu - y'_\nu w'_\mu)\,\mathrm{d}^3u'\right] = -g_{\varrho\mu} w_\nu + g_{\varrho\nu} w_\mu \,.$$

If we evaluate the left-hand side here by means of (69), we get

$$k_{\varrho\nu r} y_\mu{}^r - k_{\varrho\mu r} y_\nu{}^r = 0 \,. \tag{76}$$

Hence

$$l_\nu k_{\varrho\nu r} = 0 \,,$$

so from (69) and (70)

$$l_\mu [w_\mu, w'_\varrho] = 0 \,. \tag{77}$$

This result is strong enough for us to be able to infer the desired P. b. relations (63) without further detailed calculation. If one writes (63) in terms of normal and tangential components, it becomes

$$[w_l, w'_l] = -2w_r \delta^r - w_r{}' \delta \tag{78}$$

$$[w_l, w'^s] = (w_l \delta)^s \tag{79}$$

$$[w^r, w'^s] = w^s \delta^r + (w^r \delta)^s \,. \tag{80}$$

Equations (78) and (79) follow directly from (77). Further, (79) shows that w^s has the desired P. b. with w'_l, so from Poisson's identity it must have the desired P. b. with $[w'_l, w''_l]$. Putting in for $[w'_l, w''_l]$ its value according to (78), we see that w^s must have the desired P. b. with w'^r, namely (80).

The work of the present section shows how a Hamiltonian formulation of the field equations on general curved surfaces may be obtained from a formulation on flat surfaces with the help of a stress tensor $T_{\mu\nu}$ satisfying the basic conditions (i)–(iv), provided the extra condition (64) is fulfilled. The result is useful, because it may be easier to establish the Hamiltonian formulation on the curved surfaces by this method than by working from the Lagrangian or by directly checking the P. b. relations between the $\mathcal{H}$'s.

The extra condition (64) means that the P. b.'s $[T_{\varrho\sigma}, T'_{\mu\nu}]$ must not involve derivatives of δ higher than the first. This seems to be a necessary condition that $T_{\mu\nu}$ must satisfy in order to be acceptable as a stress tensor in a Hamiltonian theory.

7. – The Effect of Gauge Transformations.

The previous discussion is not adequate for a field theory involving the electromagnetic field, because there are then gauge transformations coming in to complicate some of the arguments. The total Hamiltonian is now

$$H=\int\{\dot{y}_\mu\mathcal{H}_\mu+\lambda\mathcal{G}\}\mathrm{d}^3u\,, \tag{81}$$

where $\mathcal{G}(u)$ is a further Hamiltonian weakly equal to zero, on the same footing as the $\mathcal{H}$'s, and $\lambda(u)$ is an arbitrary coefficient on the same footing as $\dot{y}_\mu$. Taking $\dot{y}_\mu=0$ and $\lambda\neq 0$, we get a Hamiltonian H which does not give rise to any motion of the surface, but yet causes some change in the field variables on the surface. This change is just a gauge transformation.

For consistency we must have, in addition to (6),

$$[\mathcal{H}_\mu, \mathcal{G}']\approx 0\,, \qquad [\mathcal{G}, \mathcal{G}']\approx 0\,. \tag{82}$$

From the weak equation (6) we can no longer infer the strong equation (9), since there may well be a linear function of $\mathcal{G}$ on the right-hand side of (9).

The $\mathcal{H}_\mu$ in (81) is not well-defined, if one ignores considerations of simplicity, because one can add to it any linear function of $\mathcal{G}$. Thus β_μ given by (8) is not well-defined. However, the uncertainty in the β's is weakly zero, so it does not affect their values, but only some of their P. b. relations.

Quantities of physical importance are gauge invariant. Such quantities have their P. b. with $\mathcal{G}$ vanishing weakly.

The theory of u-scalars and surface-independent and curvature-independent quantities given in § 3 still holds, provided we restrict ourselves to gauge invariant quantities, since for such quantities the uncertainty introduced into their P. b.'s with $\mathcal{H}_\mu$ arising from the uncertainty in $\mathcal{H}_\mu$ must vanish weakly.

Let us assume that β_μ is gauge invariant. Then it is reasonable to set up equation (24) to connect β_μ with $T_{\mu\nu}$. If β_μ were not gauge invariant we could not do this, because we want $T_{\mu\nu}$ to be gauge invariant.

With β_μ gauge invariant we can proceed as before and show that $T_{\mu\nu}$ is surface independent. The uncertainty in β_μ leads to $T_{\mu\nu}$ being uncertain to the extent that one may add to it a linear function of $\mathcal{G}$. This uncertainty vanishes weakly, so it does not affect the value of $T_{\mu\nu}$, but only the P. b. of $T_{\mu\nu}$ with quantities that are not gauge invariant.

RIASSUNTO (*)

Il tensore dello sforzo non è completamente determinato dalle consuete condizioni impostegli. Tuttavia, se si ha una formulazione hamiltoniana delle equazioni di campo su generiche superficie spaziali nello spazio-tempo si può connettere al tensore dello sforzo la densità di funzioni hamiltoniane, purchè questa soddisfaccia alcune condizioni di indipendenza dalla curvatura, definendo così completamente il tensore dello sforzo. Viceversa, se è data una formulazione hamiltoniana solo su superficie piane ed è anche dato il tensore dello sforzo, lo si può usare per dare la formulazione hamiltoniana su superfici generiche, purchè le relazioni delle parentesi di Poisson del tensore dello sforzo soddisfacciano determinate condizioni equivalenti all'indipendenza dalla curvatura.

(*) *Traduzione a cura della Redazione.*

Une interprétation nouvelle de la mécanique ondulatoire est-elle possible?

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Résumé. — L'auteur, après avoir rappelé la tentative qu'il avait fait en 1927, sous le nom de « théorie de la double solution » pour obtenir une interprétation causale et objective de la Mécanique ondulatoire, expose quel lui parait être l'état actuel de la question à la suite des travaux qu'il a repris dans cette direction depuis trois ans.

Aucun physicien n'ignore aujourd'hui que la Mécanique ondulatoire a reçu depuis 25 ans une interprétation « purement probabiliste » dans laquelle l'onde associée au corpuscule n'est plus qu'une représentation de probabilité dépendant de l'état de nos informations et susceptible de varier brusquement, avec elles, tandis que le corpuscule est conçu comme n'ayant pas de localisation permanente dans l'espace et, par suite, ne décrivant pas une trajectoire bien définie. Cette manière de se représenter le dualisme onde-corpuscule a reçu le nom de «complémentarité», notion assez peu précise et assez élastique que l'on a cherché à extrapoler, d'une façon parfois dangereuse, en dehors des domaines de la Physique.

Cette interprétation de la Mécanique ondulatoire, bien différente, je le rappellerai, de celle que j'avais envisagée au début de mes recherches, est due principalement à MM. Born, Bohr et Heisenberg dont les brillants travaux sont d'ailleurs dignes de la plus grande admiration. Elle a été assez rapidement adoptée par presque tous les théoriciens malgré les réserves expresses que faisaient à son sujet des esprits aussi éminents que MM. Einstein et Schrödinger et les objections qu'ils lui opposaient. Personnellement, après avoir proposé une interprétation tout à fait différente, je me suis rallié à celle qui pevenait « orthodoxe » et je l'ai enseignée pendant de longues années. Mais

depuis environ trois ans, à la suite notamment de tentatives faites par de jeunes physiciens, MM. Bohm et Vigier, je me suis à nouveau demandé si après tout ma première orientation en face du problème si ardu posé par le dualisme onde-corpuscule n'était pas la bonne. Sans entrer dans des discussions mathématiques que l'on trouvera ailleurs [1], je voudrais résumer quelle avait été ma pensée originelle sur ce sujet et de quelle manière j'ai pu, dans ces dernières années, la reprendre et la développer.

Dans mes premiers travaux sur la Mécanique ondulatoire, qui remontent à 1923, j'avais clairement aperçu qu'il fallait, d'une façon générale, associer la propagation d'une onde au mouvement de tout corpuscule; mais l'onde continue, du type de celles qui sont classiques en Optique, que j'avais été amené à considérer et qui est devenue l'onde ψ de la Mécanique ondulatoire usuelle, ne me paraissait pas décrire exactement la réalité physique: seule, sa *phase*, directement reliée au mouvement du corpuscule, me semblait avoir une signification profonde et c'est pourquoi j'avais d'abord nommé l'onde que j'associais au corpuscule « l'onde de phase », dénomination aujourd'hui bien oubliée, mais qui dans mon esprit avait sa raison d'être. Pourquoi avais-je ainsi attaché beaucoup plus d'importance à la phase de cette onde qu'à son amplitude? Il y avait à cela deux raisons. La première était que ma découverte reposait essentiellement sur une analyse, faite suivant les conceptions relativistes, du rapport qui existe entre la fréquence d'une horloge et la fréquence d'une onde. J'avais remarqué que ces deux sortes de fréquences ne se transformaient pas de même lors d'une transformation de Lorentz et que, pour cette raison, si une horloge se déplace au sein d'une onde en propagation, elle ne peut rester en phase avec cette onde que si elle a un mouvement parfaitement déterminé. Si l'on conçoit le corpuscule comme une sorte d'horloge qui doit rester en phase avec une onde qui l'environne, l'accord des phases implique une relation bien définie entre la propagation de l'onde et le mouvement du corpuscule. Appliquant cette idée au cas le plus simple, j'avais reconnu qu'à la propagation d'une onde plane monochromatique doit être associé le mouvement rectiligne et uniforme du corpuscule et, en m'inspirant, pour introduire la constante h, de la théorie des quanta de lumière (photons) d'Einstein, j'étais parvenu aux relations fondamentales

$$W = h\nu\,, \qquad \lambda = \frac{h}{p}$$

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[1] On pourra lire à ce sujet un opuscule que j'ai publié avec M. Vigier sous le titre: *La Physique quantique restera-t-elle indéterministe?* (Gauthier-Villars, 1953) ainsi que deux notes complémentaires importantes (*Compt. Rend. Ac. Sc.*, **236**, 1453 (1953) et **237**, 441 (1953)). Je compte pouvoir publier prochainement un ouvrage d'ensemble sur ce sujet chez l'éditeur Gauthier-Villars.

reliant l'énergie W et la quantité de mouvement p du corpuscule à la fréquence ν et à la longueur d'onde λ de l'onde associée. Ces relations que j'avais généralisées de diverses façons et dont j'avais tiré des conséquences intéressantes, sont restées à la base de la Mécanique ondulatoire et ont été brillamment vérifiées par la découverte de la diffraction des électrons. Mais comme elles ne faisaient intervenir, pour les relier au mouvement du corpuscule, que des éléments provenant de la phase de l'onde, j'attribuais à cette phase beaucoup plus d'importance qu'à l'amplitude de l'onde.

C'est qu'en effet, et c'est là la deuxième raison à laquelle je faisais allusion plus haut, l'amplitude continue des ondes que je considérais, en particulier l'amplitude constante des ondes planes monochromatiques, ne me paraissait pas avoir une signification physique aussi nette que la phase. Ne donnant aucune prérogative particulière à aucun point de l'espace, elle n'était pas susceptible de représenter la position du corpuscule: tout au plus, pouvait-on supposer, comme on le fit bientôt, qu'elle donnait par son carré la « probabilité de présence » du corpuscule en chaque point. Mais cette idée ne me donnait pas entière satisfaction et je rêvais d'un phénomène ondulatoire global donnant, dans le cadre de l'espace et du temps, une description unitaire du dualisme onde-corpuscule.

Cependant, les idées que j'avais semées ayant germé, les travaux d'autres savants tels que MM. Schrödinger et Born, influencés aussi par la Mécanique quantique de M. Heisenberg, faisaient faire de grands progrès à la Mécanique ondulatoire. Et il devenait de jour en jour plus évident que l'onde ψ avec son amplitude continue ne pouvait servir qu'à des prévisions statistiques: ainsi s'orientait-on peu à peu et presque inévitablement vers cette interprétation « purement probabiliste » dont MM. Born, Bohr et Heisenberg furent les principaux promoteurs et qui a triomphé depuis lors. Etonné de cette évolution qui ne me paraissait conforme ni à la mission « explicative » de la Physique théorique, ni à mes intuitions primitives, j'ai été amené à penser, vers 1925-27, qu'il y avait lieu de considérer dans tout problème de Mécanique ondulatoire deux solutions couplées de l'équation des ondes: l'une, l'onde ψ, dont la phase peut être interprétée physiquement, mais qui, en raison du caractère continu de son amplitude, n'a qu'une signification statistique et subjective; l'autre, que je nommais l'onde u, ayant même phase que l'onde ψ, mais dont l'amplitude présenterait autour d'un point de l'espace de très hautes valeurs (je disais alors qu'elle présentait une singularité mathématique en un point de l'espace) et qui, précisément en raison de cet accident local, serait susceptible de décrire objectivement le corpuscule. J'obtenais ainsi ce qu'il m'avait toujours semblé nécessaire de chercher: une image du corpuscule où celui-ci apparaît comme le centre d'un phénomène ondulatoire étendu auquel il est intimement incorporé. Et, grâce au parallélisme que postulait ma théorie entre l'onde u réalité objective et l'onde ψ construction de notre esprit, il me

semblait possible de justifier les propriétés de prévision statistique que l'on venait à juste titre d'attribuer à l'onde ψ.

Telle est l'idée qui avait germé dans mon esprit à cette époque et dont la curieuse subtilité m'étonne encore moi-même aujourd'hui. Je l'avais appelée du nom suggestif de « théorie de la double solution » et je l'avais exposée dans un article paru dans le *Journal de Physique* en Mai 1927 (²): elle représentait dans toute sa complexité ma véritable pensée. Mais, pour la commodité de l'exposé et pour éviter d'avoir à donner des justifications mathématiques difficiles, je lui avais donné (notamment dans mon exposé au Conseil de Physique Solvay d'Octobre 1927) une forme simplifiée, que je crois aujourd'hui beaucoup moins profonde. Dans cette forme, que j'avais appelée la théorie de l'onde pilote, je considérais le corpuscule comme une réalité physique donnée à priori, qui serait guidé, piloté, par la phase de l'onde ψ. Cette manière de présenter mes conceptions avait l'inconvénient, que ne présentait pas la théorie primitive de la double solution, de faire guider le corpuscule, considéré comme une réalité objective, par une onde ψ dont je reconnaissais déjà, comme tous les autres théoriciens, le caractère statistique et subjectif. C'était là, je le reconnus vite, un point de vue inacceptable. Découragé par l'accueil peu favorable fait à mes conceptions par la plupart des physiciens théoriciens que séduisaient l'élégance formelle et l'apparente rigueur de l'interprétation probabiliste, je me suis ensuite rallié pendant de longues années à cette interprétation et c'est seulement depuis 1951 que je me suis à nouveau demandé si, au fond, ma première orientation n'était pas la meilleure.

Dans mon mémoire de 1927, j'avais montré que, si la fonction d'onde u présente une région extrêmement petite où elle atteint des valeurs très élevées, ce qui a lieu par exemple autour d'une singularité ponctuelle, le fait même que l'onde u obéit à la même équation d'ondes que l'onde ψ entraîne que cette région singulière se déplacera au cours du temps en suivant l'une des lignes de courant envisagée par l'image hydrodynamique de l'onde ψ. La formule qui dans ma théorie donnait la vitesse, en chaque point de la trajectoire, du corpuscule identifié à cette région singulière prenait, quand on pouvait négliger les corrections de relativité, la forme simple suivante

$$\boldsymbol{v} = -\frac{1}{m}\,\mathbf{grad}\,\varphi\ ,$$

$\boldsymbol{v}$ et m étant la vitesse et la masse du corpuscule, φ la phase de l'onde u définie en posant $u = f \exp[(2\pi i/h)\varphi]$ avec f et φ réels. Comme par hypothèse la phase φ devait être la même pour l'onde ψ et pour l'onde u, la formule précédente que j'ai nommée « formule de guidage » permettait aussi de considérer

(²) On trouvera la reproduction de cet article dans le fascicule mentionné plus haut en note.

le corpuscule comme guidé par son onde ψ, mais, je l'ai dit, ce point de vue n'était pas réellement satisfaisant.

La formule du guidage m'avait permis de montrer que le corpuscule se trouvait ainsi obéir à une Dynamique où intervenait, à côté des forces du type classique, une force quantique dérivant d'un « potentiel quantique » et traduisant la réaction sur le corpuscule (au sens étroit du mot) du phénomène ondulatoire étendu auquel il se trouvait *incorporé* en tant que région singulière. Pour cette raison, le corpuscule n'était pas soumis uniquement, comme en Mécanique classique, aux seules forces qui s'exercent sur lui le long de sa trajectoire sans subir aucune répercussion de la présence d'obstacles qui pourraient se trouver au loin en dehors de sa trajectoire: dans ma conception, le mouvement du corpuscule incorporé à l'onde u devait subir en outre, par l'intermédiaire du potentiel quantique, l'influence de tous les obstacles susceptibles d'entraver la libre propagation du phénomène ondulatoire étendu dont il était solidaire et j'apercevais dans cette circonstance une explication possible des phénomènes d'interférences et de diffraction.

Le postulat affirmant l'identité des phases des ondes u et ψ m'avait permis de montrer, sous réserve peut-être d'une justification plus rigoureuse, que l'intensité de l'onde ψ (exprimée par la quantité $|\psi|^2$) devait mesurer la probabilité de trouver le corpuscule en un point de l'espace à un instant donné quand on ignore laquelle des trajectoires définies par la formule du guidage est effectivement suivie par le corpuscule (cette ignorance étant d'ailleurs, semble-t-il, imposée, en raison de l'existence du quantum d'action, par la nature même des observations que nous pouvons faire).

J'avais aussi, dans mon mémoire de 1927, tenté de donner une explication du succès incontestable obtenu par la Mécanique ondulatoire des systèmes de corpuscules dans l'expace de configuration développé par M. SCHRÖDINGER. A mon point de vue, chaque corpuscule du système devait être représenté comme une sorte de singularité incorporé dans un phénomène ondulatoire étendu se propageant dans l'espace physique à trois dimensions: l'espace fictif de configuration étant formé à l'aide des coordonnées, ici bien définies à chaque instant, des diverses singularités, l'onde ψ de la théorie de M. SCHRÖDINGER serait seulement une représentation dans cet espace fictif de la probabilité de localisation des corpuscules dans l'espace physique.

Toutes ces questions effleurées dans mon mémoire de 1927, je les ai à nouveau étudiées depuis trois ans en m'inspirant d'ailleurs par moments de certains résultats dûs à MM. BOHM et VIGIER. Plusieurs de mes résultats de 1927 ont pu ainsi être précisés ou améliorés et l'extension de l'idée de double solution au cas des fonctions d'onde u et ψ à plusieurs composantes, telles que celles que l'on est amené à considérer dans la Mécanique ondulatoire de l'élection à spin de M. DIRAC, peut se faire sans grandes difficultés nouvelles.

Malheureusement il existe d'autres difficultés d'une nature bien plus grave

qui avaient beaucoup contribué, il y a 25 ans, à me décourager de continuer dans cette voie. Avant d'étudier ces difficultés et les raisons qui, aujourd'hui, me font espérer qu'elles ne sont peut-être pas insurmontables, je dois d'abord exposer une idée nouvelle que j'ai introduite depuis trois ans dans la théorie de la double solution et d'ont l'importance me paraît capitale: celle que l'équation des ondes u est, en principe, non linéaire et qu'elle est, par suite, différente de l'équation linéaire admise pour l'onde ψ, bien que ces deux équations puissent être considérées comme étant presque partout dans l'espace pratiquement identiques.

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L'idée d'attribuer à l'onde u une équation non linéaire m'a été suggérée par l'analogie existant entre la conception du guidage d'un corpuscule par l'onde environnante et les résultats de MM. Georges Darmois et Einstein au sujet du mouvement d'une particule en Relativité généralisée.

En Relativité généralisée, les coëfficients $g_{\mu\nu}$ de la métrique d'espace-temps obéissent à des équations non linéaires (même dans le cas du vide) et il en est de même des quantités $\gamma_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$, différences entre les $g_{\mu\nu}$ et leurs valeurs constantes galiléennes $g^{(0)}_{\mu\nu}$. Néanmoins, mises à part de très petites régions singulières de l'espace-temps où les $\gamma_{\mu\nu}$ prendraient de très grandes valeurs et qui constitueraient dans les vues de M. Einstein les tubes d'univers des corpuscules, les $\gamma_{\mu\nu}$ obéissent approximativement à des équations linéaires. M. Einstein et ses élèves [3], retrouvant un résultat aperçu antérieurement par M. Georges Darmois [4], ont montré que les régions singulières doivent se déplacer dans l'espace au cours du temps de telle façon que le tube d'univers très délié représentant leur mouvement coïncide avec une géodésique du champ extérieur. Ce résultat est très remarquable parce qu'il permet de déduire directement des équations du champ elles-mêmes le mouvement des corpuscules sans avoir à introduire comme postulat supplémentaire (comme on le fait dans les exposés usuels de la Relativité généralisée) le fait que la ligne d'Univers d'une particule est une géodésique de l'espace-temps. Assurément les raisonnements de MM. Darmois et Einstein portent uniquement sur le champ de gravitation (et même sur le champ de gravitation non quantifié, sans gravitons), mais il est dans la ligne d'idées d'Einstein de penser que les fonctions d'ondes u relatives aux diverses sortes de corpuscules viendront un jour trouver leur place dans le cadre de la structure d'un espace-temps convenablement généralisé. Il apparaît dès lors probable que ces « champs » u, comme le champ des $g_{\mu\nu}$, devraient obéir à des équations non linéaires.

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[3] Einstein et Grommer: *Sitz. Preuss. Akad. Wiss.*, **1** (1929); Infeld: *Rev. Mod. Phys.*, **24**, 408 (1949).

[4] G. Darmois: *Les équations de la gravitation Einsteinienne* (*Mémorial des Sciences mathématiques*) (Paris, 1927).

Cherchons donc à introduire de la non-linéarité dans la théorie de la double solution. Bien entendu l'équation des ondes ψ, simples représentations fictives de probabilité, doit rester linéaire car le principe de superposition, conséquence nécessaire de la signification statistique du ψ, doit être satisfait et implique la linéarité. L'équation de l'onde ψ est celle que l'on connait bien en Mécanique ondulatoire usuelle. La théorie de la double solution suppose que, sauf dans une région très petite constituant le « corpuscule » au sens étroit du mot, l'onde u obéit à la même équation linéaire que l'onde ψ. Mais ceci n'empêche pas d'admettre que la véritable équation d'ondes de u soit une équation non linéaire, les termes non linéaires n'ayant une influence sensible que dans une très petite région de l'espace, en général mobile, où les valeurs de u deviendraient très grandes. En dehors de cette petite région singulière et peut-être aussi du bord des trains d'ondes où les dérivées de u peuvent devenir très grandes, les termes non linéaires seraient assez petits pour que l'onde u puisse obéir très approximativement à la même équation linéaire que l'onde ψ.

EINSTEIN a beaucoup insisté sur une propriété importante des équations non linéaires. Si les équations d'un certain champ sont linéaires, on peut toujours trouver une solution à singularité de ces équations telle que la singularité ait un mouvement prescrit à l'avance. On pourra d'ailleurs ajouter à la solution à singularité une solution continue et cette adjonction n'aura aucune influence sur le mouvement de la singularité. Il n'en est plus du tout de même si les équations du champ sont non linéaires car on ne peut plus alors obtenir une solution en ajoutant plusieurs solutions: la non-linéarité crée une sorte de solidarité entre des solutions qui auraient été indépendantes si l'approximation linéaire avait été valable partout.

Appliquons cette remarque à la théorie de la double solution. Nous supposons que l'onde u obéit à une équation qui est non linéaire dans la très petite région singulière, mais qui se réduit sensiblement en dehors à l'équation linéaire de la Mécanique ondulatoire. A l'extérieur de la région singulière, on peut trouver [5] une solution u_0 qui a une valeur très faible en dehors de la région singulière, mais qui croît très rapidement au voisinage de cette région et qui comporterait une singularité mathématique dans cette région si l'équation linéaire y restait valable. On peut aussi trouver une solution continue v de même phase qui est du type usuel en Mécanique ondulatoire. On aura donc à l'extérieur de la région singulière, une solution de la forme $u = u_0 + v$. Cette solution se prolongera dans la région singulière non linéaire, mais la décomposition de u en u_0 et v n'aura plus aucun sens et il est vraisemblable, conformément aux vues de M. EINSTEIN, que la fonction u ne devra présenter aucune véritable singularité mathématique dans la région singulière. Seule-

(5) Cette solution existe certainement dans le cas de l'absence de champ et aussi dans d'autres cas récemment étudiés par M. PETIAU.

ment la non-linéarité régnant dans cette région aura pour effet de rendre solidaires à l'extérieur les deux ondes u_0 et v qui seraient indépendantes si la linéarité régnait partout. Ainsi peut s'expliquer comment la région singulière définie par les grandes valeurs de u peut sembler être guidée par l'onde v et suivre ses lignes de courant. Ce guidage, exprimé par la formule du guidage, a pour effet que l'onde u dans la région singulière reste toujours en phase avec l'onde v environnante. Nous retrouvons ici, en assimilant la région singulière au corpuscule, l'image qui m'avait orienté au début de mes recherches: la très petite région singulière, constituant le corpuscule et assimilée à une horloge, se déplacerait au sein de l'onde v dont elle est intimement solidaire, de façon à rester constamment en phase avec elle.

Nous obtenons ainsi une image de l'onde u qui est beaucoup plus précise que celle que j'avais envisagée il y a vingt cinq ans. Elle peut être décrite comme comportant une « base » étendue qui est l'onde v, solution de l'équation linéaire usuelle de la Mécanique ondulatoire, au sein de laquelle se trouve greffée une sorte d'aiguille extrêmement fine constituée par les valeurs élevées de u dans la région singulière. Cette aiguille, soudée sur sa base par la non-linéarité, court au sein de l'onde v de façon à rester toujours en phase avec elle. De plus, comme l'onde ψ solution de l'équation linéaire est une construction de notre esprit, nous pouvons la définir comme partout proportionnelle à v et poser $\psi = Cv$. Comme l'onde u, et par suite l'onde v qui coïncide avec u à l'extérieur de la région singulière, est une réalité objective, elle doit avoir une valeur bien déterminée. Au contraire, nous sommes libre de « normer » l'onde ψ comme nous le voulons de façon que le carré de son module puisse jouer le rôle de probabilité de présence que tout le monde lui reconnaît. Ceci se fera par un choix convenable de la constante C. Ainsi est bien marquée la différence entre l'onde u qui décrit une réalité objective et l'onde ψ représentation subjective de probabilités. De cette analyse, l'idée primitive de la double solution sort confirmée, mais considérablement précisée et affinée.

Nous allons maintenant montrer qu'avec cet ensemble d'images, on peut lever deux difficultés très importantes qui semblaient barrer la route au développement de mes conceptions de 1927.

La première de ces difficultés provient des succès remportés par la méthode usuellement employée en Mécanique ondulatoire pour le calcul des états stationnaires d'un système quantifié (calcul des valeurs propres de l'énergie). Ce calcul s'effectue, depuis les mémorables travaux de M. SCHRÖDINGER en 1926, en déterminant les solutions (fonctions propres) de l'équation linéaire des ondes ψ qui sont uniformes et continues et satisfont aux conditions aux limites correspondant au problème envisagé. Les résultats ainsi obtenus pour les énergies quantifiées sont certainement exacts. Comment ceci est-il possible si l'onde ψ n'est qu'une fiction et si la véritable onde est l'onde u qui précisément n'obéit jamais aux conditions imposées aux fonctions propres,

puisque ou bien elle comporte une singularité (c'était mon ancienne hypothèse de 1927) ou bien elle n'obéit pas partout à l'équation linéaire usuelle (c'est mon hypothèse actuelle)? Mais ce que nous avons dit sur la structure de l'onde u nous permet maintenant d'apercevoir tout de suite la solution de la difficulté. L'onde v qui forme la base de l'onde u étant proportionnelle à l'onde ψ, ses formes stationnaires seront celles que calcule la Mécanique ondulatoire usuelle et la région singulière qui est incorporée en un point de l'onde v sera, pour une forme stationnaire de v, le siège d'un phénomène ondulatoire dont la fréquence est égale à la fréquence propre du calcul habituel. Il semble que la difficulté soit ainsi entièrement levée.

Une autre difficulté qui m'avait complètement arrêté, il y a 25 ans, paraît aussi pouvoir être écartée de la même façon. Elle est relative au mouvement du corpuscule dans les phénomènes d'interférences et peut s'exposer d'une manière particulièrement frappante sur l'exemple de l'expérience bien connue des trous d'Young. Je rappelle d'abord que l'inscription des phénomènes d'interférences ou de diffraction, par exemple sur une plaque photographique, résulte, l'expérience le prouve, de l'arrivée successive sur la plaque photographique des corpuscules incidents (photons dans le cas de la lumière). L'intensité de l'onde (le $|\psi|^2$ de la Mécanique ondulatoire) doit donc être interprétée comme mesurant la probabilité de localisation du corpuscule en chaque point; là où l'intensité est grande, il y a une forte probabilité pour l'arrivée d'un photon, là où elle est faible ou nulle, il y a une probabilité faible ou nulle pour cette arrivée. Ainsi se trouve justifié, mais d'une façon tout à fait différente de la façon ancienne, le procédé employé par l'optique classique pour le calcul des franges brillantes ou obscures. Mais ce qui reste essentiel dans cette interprétation, c'est que l'on puisse raisonner comme en optique classique pour le calcul des intensités, c'est-à-dire que l'on puisse employer des ondes continues du même type que les ondes ψ.

Voici alors la grosse difficulté que j'aperçevais naguère en réfléchissant à l'expérience des trous d'Young. Dans cette expérience, on envoie une onde lumineuse dans la direction normale sur un écran percé de deux petites ouvertures circulaires. Dans le calcul classique en Optique, on calcule l'onde à la sortie de l'écran en considérant les deux trous d'Young comme jouant le rôle de deux petites sources cohérentes de même intensité. On démontre alors aisément qu'au voisinage de l'axe de symétrie et loin de l'écran, les surfaces d'égale phase sont des ellipsoïdes et que les surfaces d'égale amplitude sont des hyperboloïdes orthogonaux à ces ellipsoïdes. On en déduit la position des franges approximativement rectilignes qui sont observables sur une plaque placée parallèlement à l'écran dans la région d'interférences. Mais, du point de vue de la théorie de la double solution, nous devons, pour décrire véritablement la réalité objective, remplacer l'onde continue classique par une onde u à région singulière. Alors, pour que la région singulière arrivant sur l'écran pé-

nètre dans l'espace situé au-delà de celui-ci, elle devra avoir passé par l'un des trous d'Young et les deux trous paraissant alors ne plus jouer du tout, comme dans le calcul classique des rôles symétriques. Je croyais à cette époque que l'onde u, se réduisant au terme u_0, devait diminuer très rapidement d'amplitude quand on s'éloignait de la région singulière et qu'elle devait tomber à de très faibles valeurs dès que l'on était à des distances macroscopiques de celle-ci: il me semblait donc impossible que les deux trous puissent jouer le rôle de deux petites sources symétriques et toute la base du calcul classique me paraissait s'effondrer, ce qui constituait une grave difficulté puisque ce calcul donne des résultats exacts et semble même seul capable de les donner.

Mais la nouvelle conception des ondes u va nous permettre d'écarter cet obstacle, l'un des plus redoutables qui paraissaient barrer la route à la théorie de la double solution. Avec cette nouvelle conception, nous devons nous représenter l'onde u incidente non pas comme étroitement concentrée autour de la région singulière, mais comme comportant une région singulière implantée sur une onde v plane monochromatique, de même forme mathématique que l'onde lumineuse classique. Dire que le photon traverse l'écran d'Young veut alors dire, conformément au simple bon sens, que la région singulière passe par l'un des trous d'Young; mais cette région singulière a des dimensions si petites (probablement inférieures à 10^{-13} cm) que, même au moment où elle traverse l'un des trous, elle n'occupe qu'une fraction infime de sa surface dont le diamètre est d'un ordre de grandeur macroscopique. On peut donc considérer que sur toute la surface des deux trous l'onde u coïncide pratiquement avec l'onde v, c'est-à-dire avec l'onde lumineuse classique. L'onde qui interfère est donc exactement proportionnelle à celle considérée dans l'explication classique avec cette seule différence que nous lui adjoignons un accident local de dimensions extraordinairement petites, la région singulière ou corpuscule (ici photon), qui court dans son sein avec la vitesse prescrite par la formule du guidage. Et ce mouvement du corpuscule a pour conséquence que la probabilité de son arrivée en un point de l'écran où s'inscrivent les franges est proportionnelle au carré de l'amplitude de l'onde classique, ce qui nous permet de retrouver le calcul traditionnel des franges malgré le caractère localement singulier de l'onde u [6].

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[6] On peut même considérer l'expérience des trous d'Young comme apportant une preuve directe de la non-linéarité de l'équation de propagation de l'onde u. En effet, v se propage comme l'onde lumineuse classique; mais, si l'équation de u était linéaire, la propagation de u_0, qui donne le mouvement du corpuscule, serait *indépendante* de la propagation de v et ne pourrait pas être influencée par l'existence du deuxième trou d'Young. Seule la non-linéarité de l'équation de u peut avoir pour conséquence de lier la propagation de u_0 à celle de v. C'est parce qu'en 1927, je n'introduisais pas la non-linéarité que l'expérience des trous d'Young me paraissait être pour la théorie de la double solution un obstacle insurmontable.

La théorie de la double solution a encore d'autres difficultés à affronter: elles concernent notamment l'étalement des trains d'ondes et la division des trains d'ondes par un miroir semi-transparent ou par l'effet d'une collision s'accompagnant de ce qu'on nomme dans l'interprétation usuelle « la réduction du paquet de probabilité ».

Disons quelques mots de la première de ces difficultés. Dans la théorie usuelle de la propagation des ondes qui repose sur des équations linéaires, les trains d'ondes, en se propageant, s'étalent dans l'espace et corrélativement leur amplitude va en s'affaiblissant. L'étude mathématique de cet étalement montre qu'il est lié au fait que, dans la théorie *linéaire* des ondes, les trains d'ondes sont représentés par des superpositions d'ondes planes monochromatiques: ces ondes planes se propagent indépendamment les unes des autres en se déphasant entre elles et il en résulte l'étalement et l'affaiblissement progressifs du train d'ondes. On peut dire que l'indépendance des composantes monochromatiques du train d'ondes amène peu à peu sa désorganisation. Au point de vue de la théorie de la double solution, il semble en résulter une conséquence très difficile à admettre physiquement. En effet, à l'extérieur de la région singulière, l'onde u doit se réduire à l'onde v qui obéit sensiblement à l'équation de propagation linéaire de la Mécanique ondulatoire; si donc l'on considère une région singulière implantée sur un train d'ondes v, la partie extérieure v de l'onde u devrait s'étaler en s'affaiblissant, et l'onde u devrait tendre à se réduire à sa partie en aiguille. En langage imagé, ceci signifierait que le corpuscule devrait perdre progressivement son onde. Peut-être cette idée n'est-elle pas à rejeter: elle me paraît cependant difficilement acceptable dans une théorie qui cherche à donner une image objective du corpuscule et qui doit rendre compte de sa permanence. Ici encore la non-linéarité pourrait peut-être nous tirer d'embarras. L'équation de l'onde u étant supposée non linéaire, cette non-linéarité doit en principe exister partout, même dans le domaine extérieur à la région singulière, domaine où les termes non linéaires sont généralement peu sensibles. Dans la région singulière, la non linéarité est prépondérante et doit avoir pour effet de souder fortement ensemble les termes u_0 et v, ce qui devrait avoir pour résultat d'empêcher le corpuscule de perdre son onde. De plus, la non-linéarité, peu sensible dans le corps du train d'ondes, peut réapparaître sur leurs bords où les dérivées de u peuvent prendre de grandes valeurs: il y a là aussi une circonstance qui peut s'opposer à l'étalement des trains d'ondes. Il apparaît donc qu'une théorie non linéaire des ondes u pourrait permettre d'obtenir des « groupes d'ondes sans étalement » représentant un corpuscule qui se déplacerait d'un mouvement rectiligne et uniforme sans perdre son onde, c'est-à-dire avec permanence du phénomène ondulatoire environnant. Des considérations que nous ne pouvons reproduire ici nous permettent de penser que l'existence de tels groupes d'ondes serait compatible avec la validité des relations d'incertitude d'HEISENBERG.

Des difficultés plus grandes encore se présentent quand on étudie par exemple la division d'un train d'ondes par un miroir semi-transparent ou par un processus de collision. Le train d'ondes u se divisant en plusieurs tronçons et la région singulière devant finalement se trouver implantée sur l'un de ces tronçons, la répétition consécutive de plusieurs processus de cet ordre semblerait encore avoir pour effet que le corpuscule perdrait peu à peu son onde: le passage à travers des écrans absorbants conduit à la même difficulté. On se trouve là, je le crois, en présence des objections les plus graves que l'on puisse faire à la théorie que j'expose ici. Des considérations de non-linéarité permettraient-elles encore de les surmonter et d'imaginer une sorte de régénération des trains d'ondes affaiblis? J'ai examiné la question, ainsi que d'autres problème difficiles relatifs à la conservation de l'énergie et aux états stationnaires des systèmes quantifiées, à la fin de l'ouvrage que j'ai l'intention de publier. J'ai cru apercevoir une possibilité de régénération des trains d'ondes provenant toujours de la non-linéarité, mais mes conclusions reposent, je l'avoue, sur des bases très fragiles et toute démonstration précise est actuellement impossible. Ce qui malheureusement rend présentement impossible une étude précise des conséquences de la non-linéarité de la propagation des ondes u, c'est que nous ne connaissons pas la forme des termes non linéaires à introduire dans les équations de propagation: nous savons seulement que ces équations doivent se réduire presque partout dans l'espace aux équations linéaires de la Mécanique ondulatoire usuelle. Les équations non-linéaires complètes ne pourraient, semble-t-il, être fournies que par une nouvelle théorie de la Relativité « supergénéralisée » qui ferait rentrer toutes les catégories de champs u dans la structure même de l'espace-temps.

* * *

En résumé, la Physique atomique contemporaine ne me paraît aucunement être arrivée à comprendre la véritable nature du dualisme onde-corpuscule qui est liée à l'existence du mystérieux quantum d'Action: elle s'est bornée à cacher son ignorance sous le mot un peu vague de « complémentarité ».

La théorie de la double solution, si l'on parvenait à la développer entièrement en l'appuyant sur des raisonnements solides, nous donnerait au contraire une image claire et satisfaisante du dualisme onde-corpuscule et, tout en respectant l'ensemble des résultats bien vérifiés de l'interprétation probabiliste actuelle, ferait comprendre les véritables causes de son succès. Elle montrerait, suivant l'affirmation maintes fois répétée par M. EINSTEIN, que la théorie actuelle est une théorie statistique parfaitement exacte, mais qu'elle n'est pas une description *complète* de la réalité physique. Si, au prix d'un effort qui serait certainement long et difficile, on parvenait à étendre la Relativité généralisée de façon à faire rentrer les ondes u des diverses sortes de

particules dans le cadre de l'espace-temps, on pourrait établir la forme des équations non linéaires satisfaites par les ondes u, étudier ce qui se passe dans les régions singulières et parvenir à comprendre la véritable nature de ces accidents spatio-temporels qui sont les corpuscules et aussi la signification profonde du quantum d'action qui est certainement lié d'une façon essentielle à la structure à la fois granulaire et ondulatoire de la matière et du rayonnement. On obtiendrait ainsi (ce n'est pas encore pour demain !) une magnifique synthèse des conceptions de la Relativité généralisée et de la théorie des Quanta.

Nous avons vu que dans la théorie des ondes u, comme d'ailleurs dans l'interprétation relativiste de la gravitation, la non-linéarité des équations de base doit jouer un rôle essentiel et seule pouvoir expliquer la solidarité de l'onde et du corpuscule. Nous sommes actuellement arrivés à l'image suivante. Un train d'onde u, constituant un corpuscule au sens *large* du mot, serait une sorte d'unité étendue et organisée, un peu analogue à une « cellule » dans l'acception biologique du terme. Il comprendrait en effet essentiellement les trois parties suivantes: 1) une sorte de *noyau*, la région singulière, le corpuscule au sens *étroit* du mot, siège de phénomènes essentiellement non linéaires; 2) une région environnante étendue, siège d'un phénomène sensiblement linéaire; 3) une enveloppe constituant les bords du train d'ondes où la non linéarité jouerait peut-être à nouveau un rôle important. Or, ce me semble être l'intervention des phénomènes non linéaires qui donnerait à cette « cellule » son unité, sa solidarité et sa permanence.

S'il est vrai que la non-linéarité soit la véritable clef de la Microphysique corpusculaire, on comprend aisément pourquoi la Physique quantique actuelle n'est pas parvenue à décrire le dualisme onde-corpuscule et a dû se contenter d'une description uniquement statistique et probabiliste des phénomènes de l'échelle atomique. Prenant à priori pour bases des équations linéaires et ne sortant pas du domaine de l'analyse linéaire, la théorie actuelle fait disparaître les accidents locaux dus à la non-linéarité (tels que les régions singulières et éventuellement les bords abrupts de trains d'ondes), elle efface ainsi les structures corpusculaires et, incapable de saisir la véritable relation entre onde et corpuscule, elle ne peut plus aboutir qu'à des images continues à caractère statistique. Si cependant, par l'emploi de l'onde ψ, elle parvient à une image statistique adéquate et à des prévisions exactes, ceci est dû, suivant ma manière de voir, au fait que la partie régulière de l'onde u, à l'extérieur des régions singulières, est étroitement apparentée, par sa forme analytique, à la forme usuelle admise pour l'onde ψ.

L'un des avantages de la théorie de la double solution serait qu'elle pourrait nous permettre de parvenir à une description et à une classification des diverses sortes de particules: la découverte incessante de nouveaux types de mésons et d'hypérons rend une semblable tâche de plus en plus urgente. Or la théorie actuelle ne parait pas pouvoir nous fournir une véritable description

des particules parce qu'elle n'a à sa disposition pour le faire que l'onde ψ à caractère statistique et les formalismes abstraits et également statistiques de la seconde quantification et de la théorie quantique des champs. On ne peut pas décrire complètement les individus quand on ne possède sur eux que des renseignements statistiques.

Et maintenant, concluons. Il se peut que j'aie tort de vouloir revenir à des conceptions plus claires que celles qui prévalent actuellement en Physique théorique. Mais je voudrais que l'on examine avec soin si ces chemins que l'on a abandonnés depuis 25 ans parce qu'on les considérait comme aboutissant à des impasses, ne seraient pas au contraire ceux qui pourraient déboucher vers la véritable Microphysique de l'avenir.

RIASSUNTO (*)

L'autore, dopo aver rammentato il tentativo fatto da lui nel 1927, sotto il nome di « teoria della doppia soluzione », per ottenere una interpretazione causale e oggettiva della Meccanica ondulatoria, espone quello che gli appare essere lo stato attuale del problema, alla luce dei lavori da lui ripresi da tre anni in questo senso.

(*) *Traduzione a cura della Redazione.*

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The Role of Eddy Diffusion in the Distribution of Ions in the Atmosphere Near the Ground.

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Summary. — On account of α-ray emission from rocks and other soil material the ionization within the first 10 cm above ground is more intense than further up. These ions are carried upward by eddy currents until they disappear by recombination. The theory of mass exchange by eddy currents was used to compute the average number of ions present at different levels above the ground assuming the validity of Schweidler's linear recombination law for small ions. The number of α-particles emitted from the ground was determined by means of a scintillation counter (photomultiplier) for various rocks, soil materials and vegetation and it was found that the contribution of ions produced near the ground and carried upward to the 100 cm level is not negligible. It may amount to nearly 10 per cent of the total number of ions present at this level.

The theory of eddy diffusion has been used extensively for the problem of finding the mean vertical distribution of nuclei, of dust particles and of the various radioactive elements in the atmosphere (1). At one time it was even believed that the compensation of the normal vertical conduction current in the atmosphere might be due to the upward transport of the positive space charge near the ground which, according to H. Ebert is caused by the prevalent adsorption of negative ions in the strongly ionized capillaries of the soil.

This possibility was quickly disproved by E. v. Schweidler (2) who showed

(1) W. Schmidt: *Der Massenaustauch in freier Luft* (Hamburg, 1925); V. F. Hess and W. Schmidt: *Phys. Zeits.*, **19**, 109 (1918); W. Schmidt: *Phys. Zeits.*, **27**, 371 (1926); J. Priebsch: *Phys. Zeits.*, **32**, 622 (1931).

(2) E. v. Schweidler: *Ann. der Phys.*, **63**, 726 (1920).

that this upward convection current amounts to about 1/100 of the normal downward conduction current. He used the improved formula for the increase of the coefficients of mass exchange with height.

It occurred to me that the theory of mass exchange could also be used to find out if the upward drift of ionized air from the immediate vicinity of the ground does contribute substantially to the total ion content of the region where most atmospheric-electric measurements are made (ion counts, conductivity, potential gradient).

The air in the first metre above ground is ionized by β- and γ-rays from the radioactive substances in the ground and in the atmosphere, by cosmic rays and by α-rays from radon, thoron and their decay products in the atmosphere. Within the first few metres from the ground the ionization produced by these sources is fairly homogeneous. Only the ionization produced by β-rays from the ground diminishes slightly with the distance from the ground. This effect was found experimentally by V. F. HESS and G. A. O'DONNELL [3]: this ionization diminishes from 1.18 I at 3 cm to 0.40 I at 100 cm from the ground.

The amount of α-ray emission from the ground itself has never been determined. In an estimate of the contributions of α-, β- and γ-rays to the total ionization of the lower layers of the atmosphere, V. F. HESS (1934) put this contribution of the α-rays from the ground tentatively as zero [4].

With the development of improved α-ray counting technique it seemed feasible to carry out experimental tests with various rock samples, samples of clay, sand, loam, humus, leaves of plants, blades of grass, etc., in order to see if this effect (on the average over land) is really negligible. Over the oceans and over water in general there is no doubt that the α-ray effect is zero. It is also zero when the soil surface is thoroughly wet or covered with snow.

Since all soil material contains traces of uranium and thorium with their decay products it is clear that at the surface of the earth a continuous formation of ions takes place within about 0 to 8.6 cm from the ground (8.6 cm is the extreme range of α-particles from natural radioactive elements).

Thus we have an α ionized layer a little less than 10 cm from the ground. These ions are carried upward by the ever present turbulent motion of the air (eddy currents). At the same time they undergo continuous recombination and the question arises whether any of these ions, produced at ground level by α-rays from the ground, are able to reach the height of 100 cm from the ground where most atmospheric electric measurements are performed.

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[3] V. F. HESS and G. A. O'DONNELL: *Journ. of Geophys. Res.*, **56**, 557 (1951).
[4] V. F. HESS: *Die Ionisierungsbilanz der Atmosphäre* (Leipzig, 1934).

Theory.

We consider a column of air of 1 cm² cross-section resting on the ground and call s the ion content of this column, *per gram of air*, as far as it is due only to the sporadic emission of α-particles by radioactive elements in the ground. s will diminish with height, on account of the recombination of ions of opposite signs. W. SCHMIDT's formula for the upward eddy diffusion current is

$$S = -A\frac{\mathrm{d}s}{\mathrm{d}h}, \tag{1}$$

where A is the coefficient of eddy diffusion (« Austausch Koefficient »). Its dimensions is $\mathrm{cm}^{-1}\,\mathrm{g}\,\mathrm{s}^{-1}$. S is the flux of ions upward from the ionized sheath of 10 cm depth to higher layers, per cm² and per second.

We consider a layer of air between the heights h and $h+\Delta h$ and calculate, for a column of 1 cm² cross-section, by which amount the total number of ions in this column will change with time (time elements $\mathrm{d}t$):

from below then enter

$$\mathrm{d}S = -A\cdot\frac{\mathrm{d}s}{\mathrm{d}h}\cdot\mathrm{d}t,$$

ions per cm² second, while

$$\mathrm{d}S' = \mathrm{d}S + \frac{\mathrm{d}S}{\mathrm{d}h}\cdot\Delta h$$

ions leave upward from the upper boundary.
Therefore remain within the column Δh

$$\mathrm{d}S - \mathrm{d}S' = -\frac{\mathrm{d}S}{\mathrm{d}h}\cdot\Delta h = +A\frac{\mathrm{d}^2 s}{\mathrm{d}h^2}\cdot\Delta h\cdot\mathrm{d}t.$$

In the column itself there are

$$s\cdot\Delta h\cdot\varrho \text{ ions} \qquad (\varrho = \text{density of air})$$

of which a certain number will disappear by recombination.

If we use Schweidler's « linear recombination law » (which holds for atmospheric air containing nuclei of condensation), we know that the rate of production of ions (q) is proportional to the actual number (n) of small ions present

$$q = \beta\cdot n^2 \tag{2}$$

where β is a coefficient which was called « coefficient of disappearance » by Schweidler and is analogous to the decay constant for radioactive substances.

The reciprocal (ϑ) of β is the « average life of an ion ».

Numerous measurements of ϑ and β have been made since and it has been found that the average life of small ions depends on the number of nuclei and on their composition. It varies between about 10 and 300 seconds for the free atmosphere. The average over sparsely populated land areas is about 60 seconds.

Since $\beta = 1/\vartheta$ is equivalent to a « decay constant » for a number of ions which disappear by recombination we see that in the column Δh considered

$$\beta \cdot s \cdot \varrho \cdot \Delta h \cdot \mathrm{d}t$$

ions will disappear by recombination within $\mathrm{d}t$ seconds.

The total change of ion content of the column Δh within $\mathrm{d}t$ seconds is

$$A \frac{\mathrm{d}^2 s}{\mathrm{d}h^2} \cdot \Delta h \cdot \mathrm{d}t - \beta \varrho s \Delta h \cdot \mathrm{d}t ,$$

which, for the stationary state must be zero. Thus we have

$$A \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \beta \varrho s .$$

The solution is

$$s = s_0 \exp \left[- \sqrt{\varrho \beta / A} \cdot h \right] , \tag{3}$$

where s_0 denotes the ion content per gram of air for $h = 0$, that is at the distance 8.6 cm from the ground.

Therefore the ion content, originating from the primarily ionized surface layer of 8.6 cm in height will diminish with altitude according to an exponential law.

Next we have to consider the change of the eddy coefficient with altitude which, according to W. Schmidt follows the formula

$$A(h) = A(1) h^{6/7} , \tag{4}$$

where $A(1)$ refers to 1 cm above ground, $A(h)$ to h cm above ground.

In the following table the values of the coefficient A (Austausch) are computed for two extreme cases: *a*) assuming $A = 50$ cm^{-1} g s^{-1} (for 100 metres above ground) and *b*) for $A = 200$, for the same altitude. These would according to W. Schmidt's formula correspond *a*) to 0.0185 and *b*) 0.0743 cm^{-1} g s^{-1} at 1 cm above ground. The table gives the increase from 10 cm to 100 cm above ground.

TABLE I. – *Coefficient of Eddy Diffusion* (A).

Height above ground	Case (a)	Case (b)
10 cm	0.13	0.53
20 »	0.24	0.97
30 »	0.34	1.37
40 »	0.44	1.75
50 »	0.53	2.12
60 »	0.62	2.48
70 »	0.70	2.83
80 »	0.79	3.18
90 »	0.88	3.51
100 »	0.96	3.86

The average of these values 10-100 cm for case (a) is 0.56, for case (b) 2.23 $cm^{-1}\,g\,s^{-1}$. It seemed sufficiently accurate for our purpose to use these mean values.

The exponent $\sqrt{\varrho\beta/A}$ computed for $=1/60$ (reciprocal of mean life of a small ion) and density of air $\varrho = 0.0012$ is

for case (a) $\sqrt{\varrho\beta/A} = 5.97\cdot10^{-3}$, for case ($b$) $2.93\cdot10^{-3}$ and the «relaxation height»,

for case (a) $= 167$ cm, for case (b) 342 cm. The «half value» heights are accordingly 116 cm for case (a) and 237 cm for case (b).

Therefore, we can expect, that, on the average more than one half of the ions produced near the ground by α-particles from the ground will escape recombination while they are carried to one metre or more by turbulent air currents.

The number of α-particles N_α emitted from the ground per cm^2 and per second can be determined experimentally with modern scintillation counters; these determinations are described below. Each α-particle produces between 1.25 and $2.7\cdot10^5$ pairs of ions. Some of them emerge from layers which are slightly below the surface, therefore it seems reasonable to assume that each produces $K=10^5$ ion pairs. Thus the number of ions produced per cm^2 surface per second within the first 10 cm above ground is $N_\alpha\cdot K=S_0$ of small

ions present at this level, we use the formulas

$$S_0 = -A\frac{ds}{dh} \quad \text{and} \quad \frac{ds}{dh} = -\sqrt{\frac{\varrho\beta}{A}}\cdot s = -\frac{S_0}{A},$$

from which we get

$$S_0 = s\cdot\sqrt{\varrho\beta A} = N_\alpha\cdot K \ ; \tag{5}$$

thus $s_0 = N_\alpha K/\sqrt{\varrho\beta A}$ which denotes the number of ions at ground level *per gram of air*. The number of ions per cm³ of air is then

$$n_0 = \varrho s_0 = N_\alpha\cdot K\sqrt{\frac{\varrho}{\beta A}}, \tag{6}$$

and at any height above the 10 cm level:

$$n_h = N_\alpha K\sqrt{\frac{\varrho}{\beta A}}\exp\left[-\sqrt{\frac{\varrho\beta}{A}}\cdot h\right]. \tag{7}$$

This formula enables us to compute the numbers of ions which may be present on account of convection (turbulence) at various heights above the lowest level where they were produced by α-rays. N_α has to be taken from counting experiments.

The Experiments.

a) *The apparatus.* – The experiments on the emission of α-particles from rocks were carried out by W. D. PARKINSON and a preliminary report has been given jointly by V. F. HESS and W. D. PARKINSON (5) at the Annual Meeting of the Amer. Geoph. Union in Washington, D.C. (May 3, 1954). The experiments were continued by V. J. KISSELBACH.

The scintillation counter consists of a photomultiplier tube No. 5819. The phosphorescent zinc sulfide is distributed evenly on the almost plane lower face of the tube which is mounted in a light-tight metal tube. This tube, magnetically shielded, forms the upper part of the housing of the preparation holder which consists of a heavy brass plate which can slide between fixed

(5) V. F. HESS and W. D. PARKINSON: *Transact. Amer. Geophys. Union*, **35** (in press).

plates, like a drawer. On the top fixed plate is a threaded hole. The end of the photomultiplier housing has been threaded to fit into this hole. In this way the phototube can be put in a predetermined position with respect to the sample. The sliding plate has a threaded hole and a circular stage that can be raised or lowered by turning. A tray containing the sample to be measured is put on top of this stage. When the sliding plate is pushed in, the movable stage and the sample tray are directly below the face of the phototube. The distance between the upper surface of the sample and the zinc sulfide layer of the phototube can be varied between 1 and 10 mm.

The turret with the photomultiplier tube contains the preamplifier and voltage divider. A shielded cable leads to the scaler (Model 1010, Atomic Instrument Company.) It contains an input strip, two decade scalers, a register and a calibrated high voltage source. The input strip contains a discriminator by means of which the minimum amplitude necessary to trip the scaler can be determined. The discriminator can be set by a control at the front of the plane. The operating voltage for the photomultiplier tube was 900 V. The counting rate naturally varies somewhat with the setting of the discriminator. Extrapolation to Zero setting gives the total number of α-particles above about 0.1 MeV.

The reproducibility of the results was tested every day by using an uranium standard.

b) *Background and Standardization.* – The great advantage of scintillation counters is their low background which enables us to detect α emission from extremely feeble sources. Spurious background pulses originate usually from active impurities either in the phosphorus or in the walls of the housing and preparation holder. Well polished brass trays give ordinarily about 0.15 counts per min. Part of it may be due to impurities in the phosphor. The background was found to be still lower (0.13 counts/min.) when the tray was filled with a pulverized sample of *Dunite* (Addie, N. Carolina) which contains only about 1/100 of the normal amount of uranium and thorium in ordinary rocks.

If finely powdered rock samples are left in the holder for a long time and then withdrawn, the background count is somewhat higher. This may be due to exhalation of radon from the sample. Even traces of the rock powder may invade the phosphorus. When the tube is rephosphored the background returns to its normal value.

The count obtained from a small sample is a maximum when the sample is directly below the face of the phosphor screen. The count will decrease if this small source (chip of uranium foil, 1 mm diameter) is moved toward the periphery of the sample holder. The dependence of the actual count upon the position of the source (radial distance from the center of the holder) was determined experimentally. Theoretical treatment of the calibration problem

was given by P. G. D. FINNEY and R. D. EVANS ([6]) and a similar calculation was carried out by W. D. PARKINSON and by NOGAMI and HURLEY ([7]). Other comparisons were made with « artificial uranium ores » consisting of a mixture of sand and uranium oxide, containing about 10^{-4} of uranium by weight. With this particular « ore » the count was 8.31 c/m. Corrected to zero discrimination setting this was 8.90 c/m while theoretically 11.6 c/m were expected. Thus the efficiency of the phosphorus indicated is 77%.

For the purpose of this paper it is not necessary to go into the evaluation of the uranium and thorium content of the various rock samples since the theory given in this paper deals only with the amount of small ions produced by a known amount of α-particles emitted per unit surface, per unit time. In order to get the emissions per cm^2 of the various rock and soil samples it is necessary to evaluate the « efficient surface » of the sample. The sample tray has a radius of 3.27 cm (area 33.6 cm^2), but from the experiments with the 1 mm^2 uranium standard at various positions (mentioned above) it was found that the counts from peripheral parts of the samples fall off rapidly with the distance from the center of the tray and were zero at 25 mm from the center. Since the count rate drops to zero when the small source is placed 25 mm from the center, and remains zero for positions greater than 25 mm, the source holder can be considered as representing an infinite area for the photo tube surface. A graphical integration of the curve illustrating the counting rate as a function of the radial distance of the source from the center gave as the « effective area » 12.9 cm^2. That is, if the whole tray is filled with any particular sample and a certain counting rate C (in counts per unit time) is obtained we get the actual counting rate per cm^2 and per unit time by dividing C by 12.9. In most cases the samples were kept at a distance of 1 mm from the phospored face of the photomultiplier tube.

Results.

The following table gives a summary of results of α-ray counts obtained with various rocks (PARKINSON) soil and vegetation samples (V. J. KISSELBACH). The background obtained either with a blank brass disc or a tray filled with *Dunite* is already subtracted in all cases. The rock samples refer to powdered rock (crushed lumps driven through 30 mesh wire gauze). The mean efficiency of the phosphorus was taken as 83% and therefore the actual counts were divided by a factor of 0.83 and then by 12.9 to express the emission in terms of α-particles per cm^2 of the effective area of the source. It is almost

([6]) G. D. FINNEY and R. D. EVANS; *Phys. Rev.*, **48**, 503 (1935).
([7]) H. H. NOGAMI and P. M. HURLEY: *Trans. Amer. Geophys. Union*. **29**. 335 (1948).

impossible to arrive at a meaningful average of these figures, taking into account the proportion of the surface of the earth of each type. The figures for dry soil seem to lie consistently between those of rocks and those of different types of vegetation. Therefore it seemed reasonable to take the mean α emission from various types of dry soil as a representative figure for the α-ray emission from the ground (land covered areas).

TABLE II. – *Number of α-particles emitted per min. (sample area* 12.9 cm^2).

I. *Acid igneous rocks*	
Di Rienzo Gneiss	2.35
Biotite Gneiss (Saxon Woods)	1.62
Garnet Schist (mean) (Mt. St. Vincent)	2.26
II. *Basic igneous rocks*	
Diorite	0.73
Pegmatite	0.04
Biotite Schist	0.79
III. *Sedimentary rocks*	0.58
IV. *Soil samples*:	
Fordham Campus (mean)	1.52
Fleetwood Top soil (mean)	1.64
Hillside Lake, N.Y.	1.0 to 1.19
Lake Placid N.Y. Topsoil	1.00
Humus, Whiteface Mt., N.Y.	0.90
Mud from Mirror Lake, N.Y.	0.27
Topsoil, Broad Mountain (Pennsylvania)	1.24
V. *Vegetation*	
Grass freshly cut	0.11
Bush leaves	1.3
Petals from various flowers	0.26 to 0.37
Dandelion leaves	0.4

Discussion.

Table II gives a general view of α emission from surface materials over land which is, of course, very incomplete especially with regard to rocks. However, it will perhaps suffice for our purpose when we consider the fact that the percentage of land surfaces covered with rocks is rather small. When we sub-

divide the results into the three classes, rocks, soil and vegetation we see that bare soil materials as dirt, loam, sand, gravel, humus are emitting less α-particles than rocks and much more than soil surfaces covered with vegetation. Samples of various types of vegetation as petals from flowers, blades of grass, leaves, pine needles are extremely low in α emission which is to be expected since these samples can have acquired radioactive materials from the ground only indirectly, through the roots. For a few hours after rainfalls there may be some α-ray activity left which is carried down from the atmosphere by precipitation. On the other hand it was found that the α emission of dry soil samples is much diminished if the sample is wetted artificially. This is to be expected on account of the absorption of the rays in the wet layer.

With dry samples of powdered rock and soil material it was found that the α counts were considerably higher when a deep dish (1 cm) was used, as compared with counts obtained with the same sample in a shallow dish (1 mm or less in height). The reason for this is the exhalation of radon from the sample. This was proved experimentally by covering the effective area of a sample with paper masks and observing the increase of α counts produced by the radioactive gases (radon, thoron) emerging and accumulating in the small volume between the sample and the lower surface of the photomultiplier tube.

Deep dish counts therefore were disregarded for soil samples and were used only for samples of vegetation where exhalation was practically zero. For soil samples total counts of several thousand were taken in most cases in order to get a statistical error of less than ± 3 percent.

The great variety in the actual counts of soil samples obtained from various points in the vicinity of Fordham University (Bronx, N.Y., Fleetwood, Tuckahoe and other points in Westchester, Dutchess and Essex County, New York, and Pennsylvania) makes it extremely difficult to choose a representative mean value for the number of α-particles emitted per second from 1 cm² of soil surface.

A glance at the table will show that a choice of a mean of 1.5 α-particles per minute from the sample (efficient surface 12.9 cm²) cannot be very wrong. This would correspond to $N = 1.94 \cdot 10^{-3}$ particles per cm², per second. If we further assume that each produces on the average $K = 10^5$ ions we obtain from formula (5), page 56,

$$S_0 = K \cdot N_\alpha = 194 \text{ ions/cm}^2 \text{ s} .$$

The number of ions actually present and produced only by α-rays from the ground follows from formulas (6) and (7); the calculations were carried out again separately for two cases:

(*a*) choosing $A = 0.56$ and (*b*) $A = 2.23 \text{ cm}^{-1} \text{ g}$.

Thus we obtain for case (*a*): $n_0 = 70$ ions/cm³ at 10 cm from the ground
$n_h = 38$ » » 110 » » » »

and for the case (*b*): $n_0 = 35$ » » 10 » » » »
$n_h = 26$ » » 110 » » » »

From this we see that at 110 cm from the ground (the usual level for ion counting) about 26-38 ions/cm actually observed have been carried upward to this level by convection.

This is almost 10% of the number of small ions found in most locations (300 to 500 ions per cm³).

In rural areas the number of small ions is about 500; 90% of them are produced «in situ» by α-, β- and γ-rays from radioactive substances in the atmosphere and by cosmic rays while about 10% are carried by convection from the ground to the one metre level. Over surfaces which are covered with vegetation the percentage of these «convectional» ions may be much less, of course.

In populated areas the number of nuclei is considerably greater than in rural areas. For populated areas it would be reasonable to use for β a value of 1/15 instead of 1/60 s^{-1}. If we carry out the computations from formulas (6) and (7), using $=1/15$ s^{-1} we get

for case (*a*): $A = 0.56$ $n_0 = 35$ I/cm³ and $n_h = 11$ I at the 100 cm level

for case (*b*): $A = 2.23$ $n_0' = 17$ » $n_h' = 10$ I » » »

Therefore we can conclude that the convection of ions produced by α-rays at ground level to the one meter level is not negligible even when the recombination of ions is increased by a factor of four. It is well to keep in mind that under these conditions the total number of small ions is reduced to about 200 per cm³.

Ackowledgment.

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RIASSUNTO (*)

A causa dell'emissione di raggi α dalle rocce ed altro materiale al suolo la ionizzazione entro i primi 10 cm al disopra del suolo è più intensa che più in alto. Questi ioni sono trasportati verso l'alto da correnti vorticose finchè dispaiono per effetto di ricombinazione. La teoria dello scambio di masse per effetto di correnti vorticose fu usata per calcolare il numero medio di ioni presenti a differenti livelli al disopra del suolo ammettendo la validità della legge di Schweidler sulla ricombinazione lineare dei piccoli ioni. Il numero di particelle α emesse dal suolo fu determinato per mezzo di un contatore a scintillazione (fotomoltiplicatore) per diverse rocce, materiali al suolo e vegetazione e fu trovato che l'apporto da ioni prodotti vicino al suolo e portati in alto al livello di 100 cm non è trascurabile. Può raggiungere circa il 10% del numero totale di ioni presenti a tale livello.

(*) *Traduzione a cura della Redazione.*

A Thermodynamic Relation between Frequency-Shift and Broadening.

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(ricevuto il 6 Novembre 1954)

Summary. — Since the nebular red-shift, on the expansion hypothesis, arises gradually on the journey by gravitational interaction with the matter that supports the geometry, the alternative suggestions for explaining the shift are not so different from the expansion hypothesis as is sometimes believed. However to discourage alternative explanations, it is pointed out here that a monochromatic beam of light traversing a medium, otherwise in thermodynamic equilibrium, cannot acquire a red-shift by interaction with the medium without suffering at the same time considerable spectral broadening. To make the latter unobservable in the direction of the beam, one would have to have recourse to an inadmissibly bizarre assumption concerning the angular distribution of the modified radiation.

1. – The Nature of the Nebular Red-Shift.

Ever since Hubble discovered in the spectra of extragalactic nebulae the cosmical red-shift, which most people put to the door of their mutual recession, an alternative explanation has been advocated time and again [1], viz. that the decrease of frequency and loss of energy occurs on the way and is caused by some kind of interaction with the medium (i.e. interstellar space) which the light traverses. Let me mention by the way, that the two views are not quite as antagonistic as they appear to be. For also on the recessional hypo-

[1] Quite recently by E. F. Freundlich and Max Born: *Gött. Nachr.* (1953), pp. 95 and 102; E. F. Freundlich, *Phil. Mag.*, **45**, 303 (1954).

thesis the change in frequency and energy ought not to be dubbed a Doppler-effect, since strictly speaking it is connected neither with the recession of the source on emission, the observer being taken to be permanently at rest, nor with the recession of the observer on arrival, the source being taken to be permanently at rest; the change is determined by the ratio of the radii of space at these two epochs, and takes place gradually on the journey owing to the gravitational interaction between the travelling light and the bulk of matter, smoothed out to uniformity in our approximate models, the matter that supports the line-element of the expanding universe. Indeed every free material particle, e.g. a cosmic ray particle, decreases its de Broglie-wave-number (or its momentum) at exactly the same rate, viz. inversely proportional to the radius of space.

But this is not the main point I wish to make here. I wish to draw attention to a general thermodynamic relation between the frequency-shift (if any) that a monochromatic ray of light suffers on traversing a given medium, by interacting with the medium, and the spectral broadening of that ray by the same interaction. The relationship between the two phenomena is such as to suggest, that the broadening would in general be the more pronounced effect. The theorem can only be proved for a medium which, apart from the test-ray, is in thermodynamic equilibrium, but it strongly suggests, so I believe, that also in other cases the two effects would not be unconnected and the broadening would usually be predominant. I do not deal in this note with the angular distribution of the light scattered with modified frequency. Hence in principle the present investigation leaves open the possibility that the shift be observed in the undeflected direction of the original ray, while the broadening manifests itself only in strongly deflected directions and thus escapes discovery. This assumption however, to say the least, would seem very artificial.

The following argument is based on the necessity of admitting, that any medium in thermodynamic equilibrium contains heat-radiation of a definite specific density u_ν, that each constituent bundle of the latter is subject to the same interaction with the medium as an external test-ray would be and therefore suffers the same modifications, and finally that the overall effect of these modifications must leave the spectral distribution u_ν unchanged. Mathematically we are faced with a one-dimensional problem of diffusion, not in space but along a straight line on which the frequency ν (ranging from zero to infinity) is plotted as abscissa. Our knowledge of u_ν amounts to knowing a static solution of the diffusion problem. This imposes an important condition on the law of diffusion which is to make the given distribution static. If the latter were constant (i.e. independent of ν) the diffusion would have to be symmetric in both directions. But to maintain, say, a negative gradient of the distribution at some point, i.e. for a given frequency, the diffusion must

favour the negative direction at that point, otherwise a positive current would tend to obliterate the gradient. Favouring the negative direction amounts to an overall red-shift at that point, i.e. for that frequency. Obviously the sign of the derivative of the static distribution indicated by the spectral distribution u_ν determines the sign of a possible frequency-shift, which will be the greater the stronger the diffusion and can only vanish, if there is no interaction that modifies the frequency.

If the energy of a portion of radiation were conserved while its frequency is modified, the spectral distribution u_ν itself would be the static solution in question. However from numerous classical considerations (e.g. reflection by a moving mirror) as well as from quantum theory we assume quite in general that whenever a portion of radiation has its frequency modified, its energy changes along with, and proportional to, the frequency. The complication of having the diffusing « something » change in amount according to the place to which it is carried, can be avoided by taking the quantity that is exchanged along the frequency-line to be not the energy, but the energy divided by the frequency. This amounts, of course, to the same as if we spoke in terms of light-quanta and their density. However, to disencumber speech in the following simple derivation, which follows a well-known pattern, we shall at first disregard the remark of this paragraph, in other words we shall formally treat the case that energy is conserved and not modified along with the frequency. Only in the result we shall replace u_ν by u_ν/ν.

2. – The Condition of Equilibrium.

Let $u_\nu \mathrm{d}\nu$ be the energy of the radiation of frequency $(\nu, \nu+\mathrm{d}\nu)$ per unit volume and $f(x)$ the fraction of this energy which during unit time (to be chosen sufficiently small for justifying our further assumptions) has its frequency increased by (algebraically) more than x. We assume that only very small increments, positive or negative, actually occur, in other words that $f(x)$ rapidly approaches to 1 for very moderate negative x, and to zero for very moderate positive x. The following consideration is virtually restricted to a near neighbourhood, of *this* order of magnitude, of a fixed frequency ν_0. Within this region we shall neglect the dependence of f on the frequency and have therefore not put it in evidence, though in the large it may be appreciable. We work out the condition that the balance of energy-transfer across the point ν_0 of the frequency-line be zero.

The transfers in the positive direction (A) and in the negative direction (B) are, respectively,

$$(1)\qquad \begin{cases} A = \int_0^{\nu_0} \mathrm{d}\nu\, u_\nu f(\nu_0 - \nu), \\ B = \int_{\nu_0}^{\infty} \mathrm{d}\nu\, u_\nu [1 - f(\nu_0 - \nu)]\,. \end{cases}$$

For the specific density we use the linear approximation:

$$(2)\qquad \begin{cases} A = \int_0^{\nu_0} \mathrm{d}\nu [u_{\nu_0} - (\nu_0 - \nu) u'_{\nu_0}] f(\nu_0 - \nu)\,, \\ B = \int_{\nu_0}^{\infty} \mathrm{d}\nu [u_{\nu_0} + (\nu - \nu_0) u'_{\nu_0}][1 - f(\nu_0 - \nu)]\,. \end{cases}$$

In A we introduce the integration variable $x = \nu_0 - \nu$, in B the variable $x = \nu - \nu_0$, both positive.

$$(3)\qquad \begin{cases} A = u_{\nu_0} \int_0^{\infty} f(x)\,\mathrm{d}x - u'_{\nu_0} \int_0^{\infty} x f(x)\,\mathrm{d}x\,, \\ B = u_{\nu_0} \int_0^{\infty} [1 - f(-x)]\,\mathrm{d}x + u'_{\nu_0} \int_0^{\infty} x[1 - f(-x)]\,\mathrm{d}x\,. \end{cases}$$

In A we have taken the liberty of extending the integrations to infinity instead of only to ν_0. All four integrals have simple meaning. E.g.

$$(4)\qquad \int_0^{\infty} x[1 - f(-x)]\,\mathrm{d}x = \frac{x^2}{2}[1 - f(-x)]\Big|_0^{\infty} - \frac{1}{2}\int_0^{\infty} x^2 f'(-x)\,\mathrm{d}x\,.$$

The integrated part vanishes. Moreover $-f'(y)\,\mathrm{d}y$, with a positive $\mathrm{d}y$, is precisely the fraction displaced by an amount between y and $y + \mathrm{d}y$. Hence the integral on the right hand side, including the negative sign, is precisely the contribution which the negative displacements yield to the mean square of the displacement. We write this as follows

$$(5)\qquad \int_0^{\infty} x\,[1 - f(-x)\,\mathrm{d}x = \tfrac{1}{2}\,\overline{x^2}_-\,.$$

On evaluating the other three integrals in a similar way and introducing similar abbreviations we obtain

$$(6) \qquad \begin{cases} A = u_{\nu_0} \bar{x}_{+} - u'_{\nu_0} \tfrac{1}{2} \bar{x}^2_{+} , \\ B = - u_{\nu_0} \bar{x}_{-} + u'_{\nu_0} \tfrac{1}{2} \bar{x}^2 . \end{cases}$$

By equating these two quantities we obtain the condition for zero transfer across ν_0:

$$(7) \qquad \bar{x} = \tfrac{1}{2} (\log u_{\nu_0})' \overline{x^2} ,$$

The horizontal bars now indicate plain averages.

If the spectral distribution u_{ν_0} is to remain unchanged by the interaction of the radiation with the medium, the condition (7) must be fulfilled for every ν_0. Inversely, if it so is, the spectral distribution remains unchanged. For if the energy with frequencies $< \nu_0$ is constant and the same holds for the energy with frequencies $< \nu_0 + \mathrm{d}\nu_0$, the energy between these limits, viz. $u_{\nu_0} \mathrm{d}\nu_0$, must also remain constant.

Up to here we have dealt with a rather fictitious case, assuming that a portion of radiation that has its frequency displaced by the interaction makes its appearance at the modified frequency with the same amount of energy. To implement the more realistic assumption explained at the end of section **1**, one has to repeat the consideration that led to the condition (7), but replacing u_{ν_0} by u_{ν_0}/ν_0. One must not be worried by the fact that one is now no longer dealing with energy balances. To soothe one's soul one may now call them photon-balances. Surely, if the spectral distribution is to remain constant with time, the number of photons within any delimited region of frequencies must also not change with time. I should particularly emphasize, that the function $f(x)$ and therefore the mean-values of the displacement and of the square of the displacement have exactly the same meaning as before. For $f(x)$ denoted that fraction of a nearly monochromatic energy portion that is displaced in a particular way. And with monochromatic radiation the energy fraction and the fraction of the number of photons is the same thing.

So on what I called the more realistic assumption, equation (7) has to be replaced by

$$(8) \qquad \bar{x} = \tfrac{1}{2} (\log u_\nu/\nu)' \overline{x^2} .$$

The subscript zero has been dropped, since the relation must hold for every frequency. The physical dimensions are all right since x is a frequency-displacement and the dash means a derivative with respect to ν. That the argument of the logarithm is not dimensionless is an irrelevant cosmetic defect

That both $\bar{x}$ and $\overline{x^2}$ must be proportional to the same, viz. to the first power of the small time unit chosen, is an old story, too well known from Brownian movement and similar cases for being called a paradox.

3. – A Quantitative Estimate.

In trying to estimate the relevances of the linear average, which is the shift, and the broadening, which is

$$\sqrt{\overline{x^2} - (\bar{x})^2}\,, \tag{9}$$

we use the spectral distribution for empty space, assuming our medium to be so rarefied that we may take its index of refraction to be 1. We are chiefly interested in a possible red-shift, which occurs, when the derivative in (8) is negative. In what follows we consider only this sign.

So we take

$$\frac{u_\nu}{\nu} = \text{constant} \cdot \frac{\nu^2}{e^{a\nu} - 1}\,, \qquad \left(a = \frac{h}{kT}\right), \tag{10}$$

$$\frac{1}{a}\left(\log \frac{u_\nu}{\nu}\right)' = \frac{2}{a\nu} - \frac{1}{1 - e^{-a\nu}}$$

$$= \frac{2}{a\nu} - 1 - \frac{1}{e^{a\nu} - 1}\,.$$

With $a\nu$ positive, it is easy to see that this is always algebraically greater than -1, the value approached asymptotically for very high frequency. Hence

$$\frac{1}{2}\left(\log \frac{u_\nu}{\nu}\right)' > -\frac{h}{2kT}\,. \tag{11}$$

It follows that the absolute value of the reciprocal of the left hand side, for which we introduce the notation

$$x_\nu = \left|\frac{1}{2}\left(\log \frac{u_\nu}{\nu}\right)'\right|^{-1} \tag{12}$$

is even for very low temperature a large number ($\gtrsim 10^{11}\ \mathrm{s}^{-1}$ for $T = 1\ ^\circ\mathrm{K}$). Now from (8) and (12)

$$|\bar{x}|\, x_\nu = \overline{x^2}\,. \tag{13}$$

A glance at (9) shows that $|\bar{x}|$ cannot be greater than x_ν. If it is considerably

smaller then the mean displacement square is considerably larger than the square of the mean displacement. One would, of course, like to have a lower bound for the quantity (9). But the linear approximation we have used for the spectral distribution is for that purpose not good enough. Formally it would allow (9) to vanish, which is, of course, impossible.

Though our whole argument refers only to a medium in thermodynamic equilibrium and could even in this case be obviated, as we mentioned at the outset, by a strange ad hoc assumption concerning the angular distribution of the modified radiation, yet our result seems to discourage the attempt to interpret the nebular red-shift in a fashion different from what is usually accepted. It is true that also in the expanding universe the red-shift of a beam of light occurs gradually on the journey and is due to (gravitational) interaction with the medium. But the latter is not in thermic equilibrium. Its average temperature decreases, a process of which the red-shift forms part. That is why the inference drawn in this note is not applicable to that case.

RIASSUNTO

Dato che lo spostamento verso il rosso delle righe spettrali delle nebulose, secondo l'ipotesi dell'espansione, ha origine gradualmente lungo il percorso per interazione gravitazionale con la materia che determina la geometria, altre interpretazioni di tale spostamento non sono poi così dissimili dall'ipotesi dell'espansione come talvolta si crede. Tuttavia, per scoraggiare interpretazioni diverse, si mostra in questo lavoro che un fascio di luce monocromatica che attraversa materia in equilibrio termodinamico, non può spostarsi verso il rosso per interazione con la materia senza subire allo stesso tempo un allargamento considerevole della riga. Per far sì che quest'ultimo non sia osservabile nella direzione di propagazione del fascio, si dovrebbe ricorrere ad una ipotesi bizzarra ed inammissibile per la distribuzione angolare della radiazione modificata.

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Quantum Theory of the Emission Process.

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Summary. — It is shown, that the emission process of a photon during the transition of an atom between two quantum states can be described in closed form by a superposition of stationary states of the total system, consisting of the atom coupled by the interaction to the system of field oscillators. In this treatment all physical quantities involved remain finite and no improper, quadratically non-integrable functions occur. The theory leads only in first approximation to the older expressions and reveals more details, beyond the domain of classical field concepts, which become of some importance in the immediate neighbourhood of resonance.

1. – Introduction.

It is well known, that the classical expression for the emission of a wave train by an electric oscillation, given by Hertz' solution

$$Z = \frac{z}{r} \cdot \exp\left[i(k_0 - i\gamma)\cdot(r - ct)\right]; \qquad r < ct, \tag{1}$$

has no proper spectrum, i.e. it cannot be represented by an ordinary Fourier integral. As a matter of fact, the same is true even for the physically unreal abstraction of a monochromatic plane wave. The integral, which represents (1) in momentum space is essentially a complex integral

$$Z = -\frac{Z}{\pi r}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp\left[-i\bar{k}ct\right]}{\bar{k} - k_0 + i\gamma}\,\mathrm{d}\bar{k}\int_{-\infty}^{+\infty}\frac{\sin \bar{k}r}{k - \bar{k}}\,\mathrm{d}k\,, \tag{2}$$

the path of which has to be appropriately chosen. Any complex path of integration implies, however, complex frequencies and, therefore, cannot be

interpreted in terms of a spectral analysis by means of any realisable spectral apparatus. If we try to interpret (2) along the real axis of k, we are led to define $1/(k-\bar{k})$ by a quadratically non-integrable improper function, the Schwartz' distribution

$$1/(k-\bar{k}) = 2i\pi\delta_+(k-\bar{k}) = i\pi\delta(k-\bar{k}) + P\,1/(k-\bar{k}) \tag{3}$$

(P = principal value).

The difficulty arising in (2) and (3) derives from the fact, that we cannot attribute a finite probability to any configuration of photons of given wavelength k and frequency $\bar{k}$.

We have concluded from this fact, in an earlier paper [1] that only the external parts of (3) can correspond to physical reality, while the parts close to the singularity have to be modified by a more detailed treatment of the involved physical problem.

We shall show, in what follows, that it is sufficient for our purpose, to consider a finite system in which a mechanical oscillator, represented by two states only, is coupled to a finite set of discrete electric dipole vibrations in an appropriately chosen cavity of finite size with perfectly reflecting walls. We neglect, in this treatment, waves of very high frequency, virtual transitions to higher states of the mechanical system and, hereby, all configurations in which more than one photon are present simultaneously. We suppose, that these omitted configurations do not appreciably influence the emission process as long as we restrict ourselves to the weak coupling case only.

2. – The Coupled System.

We shall denote by k_0 the unperturbed frequency of our mechanical oscillator and by c_0 its probability amplitude in the excited state (oscillator excited, no photon present) and by

$$k_1\,,\quad k_2\,,\ldots,\quad k_k\,,\ldots$$

$$c_1\,,\quad c_2\,,\ldots,\quad c_k\,,\ldots$$

the unperturbed frequencies and probability amplitudes of the field states (oscillator in ground state, one photon present).

Establishing an interaction between the mechanical oscillator and each individual electromagnetic vibration, the unperturbed frequency spectrum of

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[1] G. Beck: *Compt. Rend.*, **236**, 465 (1953).

the total system will become shifted and the new frequency spectrum becomes

$$\bar{k}_0\,,\quad \bar{k}_1\,,\quad \bar{k}_2\,,\quad \ldots,\quad \bar{k}_k\,,\quad \ldots\,.$$

To each value $\bar{k}_l$ belongs a normal vibration of the total system, a collective motion in which in general both the mechanical oscillator and all considered electromagnetic vibrations will participate.

The l-th normal vibration will be determined by a set of equations

$$(4)\qquad \begin{cases} (k_0 - \bar{k}_l)\cdot c_0^l + \alpha_{01}\cdot c_1^l + \alpha_{02}\cdot c_2^l + \ldots = 0 \\ \alpha_{10}\cdot c_0^l + (k_1 - \bar{k}_l)\cdot c_1^l = 0 \\ \alpha_{20}\cdot c_0^l + (k_2 - \bar{k}_l)\cdot c_2^l = 0 \\ \cdots\cdots\cdots \end{cases}$$

The α_{0k} and α_{k0} represent the coupling constants (matrix elements) of our system. For the moment they only have to satisfy the requirement of forming a hermitian matrix. $\bar{k}_l$ is the l-th root of the determinant of (4).

The simple structure of (4) facilitates greatly the discussion. We conclude immediately from (4):

a) The probability amplitudes c_k^l satisfy both the orthogonality relations

$$(5)\qquad \sum_k c_k^{l*}\cdot c_k^{l'} = \delta^{ll'}$$

and the completeness relations

$$(6)\qquad \sum_l c_k^{l*}\cdot c_{k'}^{l} = \delta_{kk'}$$

if these quantities are normalised in the usual way to unity.

b) For the perturbed frequencies $\bar{k}_l$ the following relation holds

$$(7)\qquad k_k - \bar{k}_l \neq 0 \quad \text{for} \quad k \neq 0\,.$$

According to (7) no crossing of the eigenvalues is possible whatever may be the strength of the coupling constants. In other words

$$(8)\qquad k_{l-1} < \bar{k}_l < k_{l+1}\,.$$

3. – The initial Conditions.

The c_k^l are the coefficients of the eigenfunction of the l-th vibration of the total system expressed in terms of linear combinations of the eigenfunctions

of the uncoupled system. We shall not write explicitly these eigenfunctions but consider the time dependent probability amplitudes

$$C_k^l(t) = c_k^l \cdot \exp\left[-i\bar{k}_l \cdot ct\right]. \tag{9}$$

Instead of considering the single normal vibrations of the system to which (9) refers we can consider wave packets i.e. superpositions of normal vibrations to which more involved time dependence corresponds. This can be done by means of an arbitrary unitary transformation S_r^l,

$$\sum_l S_r^l \cdot C_k^l(t) = D_{rk}(t) \tag{10}$$

which describes the time development of the probability amplitude of the k-th state in the r-th wave packet. Choosing the unitary transformation S_r^l appropriately, we can assure, for $t = 0$, any given initial configuration.

If we choose, in particular,

$$S_r^l = c_r^{l*} \tag{11}$$

we find according to (9), (10) and (6)

$$D_{rk}(t) = \sum_l c_r^{l*} \cdot c_k^l \cdot \exp\left[-i\bar{k}_l \cdot ct\right], \tag{12}$$

$$D_{rk}(0) = \delta_{rk}. \tag{13}$$

The unitary transformation (11) transforms, therefore, for $t = 0$, the uncoupled individual vibrations to principal axis.

In particular, $D_{0k}(0)$ describes the configuration, in which initially only the excited mechanical oscillator is present and no photons exist in the field. $D_{00}(t)$ describes, therefore, the damping of the excited oscillator state, while $D_{0k}(t)$, with $k \neq 0$, describes the emission of the field.

Expressions (12) can still be considerably simplified. Replacing in (5) for c_k^l the values which we find from (4),

$$c_k^l = \frac{\alpha_{k0}}{\bar{k}_l - k_k} \cdot c_0^l \tag{14}$$

we obtain

$$c_0^{l*} \cdot c_0^l = \left\{1 + \sum_r \frac{\alpha_{0r}\alpha_{r0}}{(\bar{k}_l - k_r)^2}\right\}^{-1}. \tag{15}$$

From (12), with (14) and (15), we find for the quantities in which we are interested

$$D_{00}(t) = \sum_l \left\{ 1 + \sum_r \frac{\alpha_{0r}\alpha_{r0}}{(\bar{k}_l - k_r)^2} \right\}^{-1} \cdot \exp[-i\bar{k}_l ct], \tag{16}$$

$$D_{0k}(t) = \sum_l \frac{\alpha_{k0}}{\bar{k}_l - k_k} \left\{ 1 + \sum_r \frac{\alpha_{0r}\alpha_{r0}}{(\bar{k}_l - k_2)^2} \right\}^{-1} \cdot \exp[-i\bar{k}_l ct], \tag{17}$$

being, for $k \neq 0$,

$$D_{0k}(t) = i \cdot \alpha_{k0} \cdot \exp[-ik_k ct] \cdot \int_0^t D_{00}(t) \cdot \exp[ik_k ct] \cdot c\,dt\,. \tag{18}$$

4. – An Approximate solution of (16).

Relations (16) and (17) reduce the solution of our problem to the determination of the shifted frequencies

$$\bar{k}_l = \bar{k}_l(k_0, k_1, \ldots; |\alpha_{01}|, |\alpha_{02}|, \ldots)\,.$$

This problem cannot be solved in all generality. We can, however, draw a few general conclusions from (16) and (17).

Due to the reflecting walls of our cavity of finite size, $D_{00}(t)$ will be a complicated function of time. Even if the frequency levels of the cavity are equidistant, as we shall assume further on, (16) does not represent a periodic function, due to the level shift introduced by the atom, but rather an almost periodic phenomenon. If we choose, however, the dimensions of the cavity large compared to the coherence length of the emitted wave train, we obtain conditions which approximate, during a certain time, the aperiodic emission in free space. The case of an atom radiating in free space is determined by (16) as a limiting process, letting our finite cavity become infinitely larger in all directions. This limiting process is, however, though uniquely determined by our procedure, a rather involved operation. The difficulties encountered by previous theories were due to the fact, that, considering only the limiting case of free space, they tried to determine the limiting value by guessing rather than by physical considerations.

We shall now assume a spherical cavity of radius R, large compared to the coherence length $1/\gamma$. Then the unshifted frequencies will be (asymptotically) equidistant,

$$\Delta k = k_r - k_{r-1} = \frac{\pi}{R}$$

and we shall assume the mechanical harmonic oscillator placed at the center of our sphere.

In order to find an approximate solution of the eigenvalue problem, we ask first whether or not we can restrict the l-th vibration to two uncoupled states only, one state of the field and the state of the excited mechanical oscillator, i.e. c_l^l and c_0^l. In this case, we find immediately from (4)

$$\bar{k}_l = \frac{k_l + k_0}{2} + \sqrt{\left(\frac{k_l - k_0}{2}\right)^2 + |\alpha_{0l}|^2} \cong k_l + \frac{|\alpha_{0l}|^2}{k_l - k_0} . \tag{19}$$

However, before adopting (19), we have to make sure that in this case the neighbouring frequencies $k_{l\pm1}$ do not contribute appreciably to the collective motion. This is the case only, if

$$\bar{k}_l - k_l \ll \Delta k . \tag{20}$$

Condition (20) can always be assured, even in the limit $R \to \infty$, at sufficiently large distance from resonance

$$k_l - k_0 \gg |\alpha_{0l}|^2/\Delta k \cong \gamma . \tag{21}$$

There exists, however, a domain

$$k_l - k_0 \leqslant |\alpha_{0l}|^2/\Delta k \cong \gamma , \tag{22}$$

which determines the region of the natural line breadth.

In the resonance domain (22) the stationary vibrations of the total system are found to be collective, but this collective motion does not affect the total domain of the line breadth, but only a small fraction of it. In this case we have

$$\bar{k}_l - k_l \sim \Delta k . \tag{23}$$

We find a first approximation of (16) if we forget, for a moment, about the resonance region (22) and introduce in (16) the asymptotic value (19). Neglecting small terms, we find

$$D_{00}(t) \cong \sum_l \frac{|\alpha_{0l}|^2 \cdot \exp[-i\bar{k}_l ct]}{(k_l - k_0)^2 + O(|\alpha_{0l}|^2/\Delta k^2)} . \tag{24}$$

Writing

$$O(|\alpha_{0l}|^2/\Delta k^2) = \gamma_l^2$$

and assuming that in a sufficiently wide region of the spectrum the γ_l do not depend on l, $\gamma_l = \gamma$, we find in the limit $R \to \infty$ with $\bar{k}_l \to k_l$ the usual result of the older theories

$$D_{00}(t) \cong \frac{1}{\pi} \int_0^\infty \frac{\gamma \cdot \exp[-ikct]}{(k - k_0)^2 + \gamma^2} \, dk \cong \exp\left[-ik_0 ct - \gamma c|t|\right], \tag{25}$$

and, according to (18), $t > 0$,

$$D_{0k}(t) \cong \frac{\alpha_{k0}}{k_u - k_0 + i\gamma} \left\{ \exp\left[-i(k_0 - i\gamma)ct\right] - \exp\left[-ik_k ct\right] \right\}. \tag{26}$$

It would be of considerable interest to find a rigorous solution of (16), which, as we shall see later, will have a more complex structure than (25), but, so far, I have not succeeded in solving this problem.

5. – The Associated Fields.

It is well known that quantum theory does not permit one to attribute to a quantum transition a finite, observable field nor a corresponding charge and current development. Only the quadratic expressions of the radiated field such as energy and momentum are observables. We can, nevertheless, ask to what extent it is possible to describe a quantum transition in classical terms.

Be u_e and u_g the time independent eigenfunctions of our mechanical oscillator in the excited and in the ground state and be Γ_0 the eigenfunction of the radiation field in the ground state, Γ_k the eigenfunction where the k-th photon is present. Then, the eigenfunction which describes the non-stationary development of the total, coupled system is given by

$$\Phi_0 = u_e \cdot D_{00}(t) \cdot \Gamma_0 + u_g \cdot \sum_{k \neq 0} D_{0k}(t) \cdot \Gamma_k \, . \tag{27}$$

Trying to form with (27) eharge and current densities we cannot obtain simple expressions of transition densities which belong to the emission of a photon as in classical theory.

On the other hand, we can consider, in (27), the quantity

$$\sum_{k \neq 0} D_{0k}(t) \cdot \Gamma_k \tag{28}$$

as defining the emitted field. To (28) we can immediately associate a classical field which, acting on an atom (test body) produces the same matrix elements

and effects as the quantum field. The concept of the associated field is useful, because it gives a direct, though not completely rigorous picture of the space-time development of the quantum field (*).

In order to construct, in our case, the associated field, we consider, in a sphere of sufficiently large radius R, the proper vibrations of the free field, limiting our attention, for our purpose, to electric dipole vibrations only, say in the z-direction, given by their respective Hertz' vectors

$$Z_k = \sqrt{\frac{3hc}{2\pi k_k R}} \frac{\sin k_k r}{k_k r} \exp\left[-ik_k ct\right] . \tag{29}$$

This is normalised (asymptotically, i.e. for a sufficiently large sphere) in order to represent one photon of energy $k_k \cdot h/2\pi$ and vanishes for $r = R$.

The coupling constants of (4) become in this case, for a harmonic mechanical oscillator of frequency k_0

$$\alpha_{0k} = \alpha_{k0} = \sqrt{\frac{3}{2}\frac{a}{R} k_0 \cdot k_k} , \tag{30}$$

where $a = e^2/mc^2$ is the classical electron radius. They have a simple physical meaning and represent the Fourier coefficients of the electric moment distribution in the emitting mechanical system during the quantum transition.

We define now the field associated with our process by

$$Z_{ass} = \sum_{k,l} \frac{k_k}{\bar{k}_l} Z^l_{0k} , \qquad k \neq 0 \tag{31}$$

with

$$Z^l_{rk} = c^{l*}_r \cdot c^l_k \cdot Z_k \cdot \exp\left[i \cdot (k_k - \bar{k}_l) \cdot ct\right] . \tag{32}$$

The physical meaning of (32) is immediately clear: it represents for the r-th wave packet the intensity with which the k-th photon contributes to the l-th vibration of the coupled system, with the common frequency $\bar{k}_l$.

The factor $k_k/\bar{k}_l$ in (31) is due to the fact, that we have to consider not the Hertz' vectors, but rather the potentials of the different free vibrations to

(*) Since the concept of the associated field does not appear in the literature and seems not to be commonly known, it shall be discussed in more detail at another opportunity. To the present author it became known during discussions and common work with J. A. Balseiro, several years ago.

be additive. In our case ($r = 0$), we can use with good approximation

$$k_k/\bar{k}_l \cong k_k/(k_0 - i\gamma)$$

as long as we deal with conditions which do not differ sensibly from the classical value (2).

The coefficients of (32), $c_r^{l*}\ c_k^l$ are the probability amplitudes for finding the k-th photon with the l-th frequency. It follows immediately from our construction, that these coefficients are square integrable. The value of the square integral of the coefficients of (31) is smaller than unity, because it does not contain the configuration in which the mechanical system is excited and no field is present. The square integrability of the coefficients assures that the spectrum of (31) is a proper spectrum, contrary to the spectrum of (2). This proves, at the same time, that the classical expressions (1) and (2) cannot be rigorously correct.

We shall now examine in more detail the associated field (31). Making use of relations (32), (29), (14) and (15) we find

$$(33) \qquad Z_{\text{ass}} \cong \sum_{l,k\neq 0} \frac{\alpha_{k0}}{\bar{k}_l - k_k}\left(1 + \sum_r \frac{\alpha_{0r}\alpha_{r0}}{(\bar{k}_l - k_k)^2}\right)^{-1} \sqrt{\frac{3hc}{2\pi k_k R}} \cdot \frac{1}{k_0 - i\gamma} \frac{\sin k_k r}{r} \exp[-i\bar{k}_l \cdot ct].$$

The double sum (33) corresponds closely to the double integral (2), without containing, however, any divergent term.

We mention here, without giving the proof explicitly, that the associated field (33) obeys Maxwell's equations for free space, outside the charge and current distributions of the emitting atoms (*).

We shall show now, that in the approximation given in § 4 (33) leads back to the classical expression (1). Using (26) the summation over l can immediately be carried through and leads to

$$(34) \qquad Z_{\text{ass}} \cong \sum_{k\neq 0} \frac{\alpha_{k0}}{k_k - k_0 + i\gamma} \sqrt{\frac{3hc}{2\pi k_k R}} \frac{1}{k_0 - i\gamma} \frac{\sin k_k r}{r} \cdot \{\exp[-i(k_0 - i\gamma)ct] - \exp[-ik_k ct]\},$$

which, transformed into an integral, is essentially equivalent to (1).

(*) Strictly speaking this statement is valid only approximately neglecting terms of the order of the contribution of negative frequencies. See: G. BECK: *Ciencia e Investigación*, **6**, 573 (1950).

It is more instructive, however, to carry out, in (33) first the summation over k, observing (30)

$$\sum_{k\neq 0}\sqrt{\frac{9hca}{4\pi R^2}}\,k_0\cdot\frac{1}{(k_0-i\gamma)(\bar{k}_l-k_k)}\,\frac{\sin k_k r}{r}\,.$$

This summation corresponds to the integration over the distribution (3) in (2). Starting from the identity

$$\frac{1}{\sin \bar{k}R}\,\frac{\sin \bar{k}(r-R)}{r}=\sum_{k\neq 0}\frac{1}{R}\left\{\frac{1}{\bar{k}-k_k}-\frac{1}{\bar{k}+k_k}\right\}\frac{\sin k_k r}{r}\,, \tag{35}$$

which can be easily verified by Fourier development, we obtain, omitting the negligible second term on the right hand side of (33)

$$\sum_{k\neq 0}\frac{1}{R}\,\frac{1}{\bar{k}_l-k_k}\cdot\frac{\sin k_k r}{r}\cong\frac{\sin \bar{k}_l(r-R)}{r\cdot\sin \bar{k}_l R}\,. \tag{36}$$

This differs essentially from the result which we would obtain by using the distribution (3) or any part of it. The main difference consists in the appearance of the phase

$$\delta_l=\bar{k}_l\cdot R\equiv\pm(\bar{k}_l-k_l)\cdot R\,,$$

which does not appear in the distribution (3) and which tends to a finite value, even in the limit $R\to\infty$.

6. – The Rigorous Treatment of the Problem of Field Propagation.

We shall now turn back to the rigorous formulation of our theory. The inconsistencies which remain in § 5 are essentially due to an inconsistency in the underlying assumptions. In order to compare our theory with previous ones, we had necessarily to refer to classical field concepts, outside of their proper domain of applicability. We have seen how the emission process depends on the coupling between the source and the field. What we have omitted in constructing our associated field is the fact, that the same mechanism which determines the excitation of the field by the source is essential also, as soon as we want to observe the emitted field by means of a real test body, i.e. by means of a second oscillator or atom.

In order to obtain a consistent treatment of the fundamental problem of field propagation in space and time we have, therefore, to couple the free field

vibrations with two mechanical oscillators placed at a certain distance from each other, the one excited (source) and the other in the ground state (test body). In order to describe this initial configuration, we have to determine first, as in § 2, the proper vibrations of the total coupled system and to construct, afterwards, a wave packet, which represents the initial configuration. The formulae which we obtain in this case, are essentially the same as (12) and (13). Only later on, when we want to eliminate the coefficients c_k^l, we meet greater mathematical difficulties which will have to be overcome. We shall not attempt to solve these difficulties in the present paper.

We can, however, discuss the properties of the solution even without having worked them out explicitly. If we denote by s and t the uncoupled states in which respectively the source and the test body are found to be excited and no field is present, we have only two probability amplitudes to consider:

$$D_{ss}(t)$$

which describes the damping of the source and

$$D_{st}(\boldsymbol{r}, t)$$

which determines the excitation of the test body as a function of time and of its relative distance from the source $\boldsymbol{r}$. The determination of $D_{st}(\boldsymbol{r}, t)$ corresponds to a physically realisable field measurement and accounts simultaneously for the finite reaction both of the source and the test body on the field. It is this reaction which is ignored in classical electrodynamics and which conduces to difficulties of quantum electrodynamics. Quantum theory permits, therefore, to solve the old conceptual problem of the role of the finite test body in the definition of a field.

7. – Conclusions.

We have seen above, that quantum theory permits us to formulate the problem of the emission process during a so-called quantum transition in an unambiguous way which leads only in first approximation to the classical expressions. So far, we have not been able to obtain explicit expressions for the higher approximations and new methods will have to be developed in order to solve equation (16) and the corresponding problem of § 6.

Still, we can already conclude, that the classical expressions (1) and (2) cannot be valid rigorously. The field emitted during a quantum transition can certainly not be represented by a single pole in the complex plane. There exist, therefore, in the domain of the natural line breadth, new and

observable physical phenomena which are not accounted for by classical theory. One phenomenon in this sense has been observed several years ago (2) and has, as a matter of fact, conduced to the present theory. It is not impossible, that the phenomenon observed by LENNUIER will be found to be included in the higher order approximations of the present theory. It would be desirable that this phenomenon be verified by other, independent experiments.

From the formal point of view, the present theory shows at least in one easily accessible special case, that there is no room, in physics for improper functions or distributions. Quantum electrodynamics uses improper functions as essential elements. The present theory may be helpful in other cases too in removing them.

The most important conclusion of the present considerations, however, is the fact, that a quantum theory of energy transfer between distant systems has to revise even the apparently simple problem of field propagation in space and time, beyond the limits of classical concepts.

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(2) R. LENNUIER: *Ann. de Phys.*, **2**, 233 (1947).

RIASSUNTO (*)

Si dimostra che il processo di emissione di un fotone durante la transizione di un atomo tra due stati quantici si può descrivere in forma chiusa per mezzo di una sovrapposizione di stati stazionari del sistema totale formato dall'atomo accoppiato col sistema degli oscillatori di campo. In questo trattamento tutte le grandezze fisiche interessate restano finite e non si presenta alcuna funzione impropria non integrabile quadraticamente. La teoria conduce solo in prima approssimazione alle espressioni già note e rivela nuovi dettagli, oltre i concetti classici di campo, di qualche importanza nell'immediata vicinanza della risonanza.

(*) *Traduzione a cura della Redazione.*

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The Behavior of Finite Particles in Conformal-Covariant Weak Field Theory.

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Summary. — A finite classical particle is shown to be associable with the new concept of particle implied by the use of the conformal sphere geometry of space-time. The introduction of the finite particle is not only Lorentz but also conformal-covariant. The mass-energy relation for these free finite particles has an extra term generalizing the Einstein special relativistic mass-energy relation for point particles. In the conjugate (position) picture a similar extra term occurs in the Einstein proper time-time relation for point particles which extends the definition of proper time to finite particles. In § 1 conformal form-invariant field and motion equations for the weak field case are given. The Maxwell Theory in vacuo is a special case. The finite effects are shown to have an electromagnetic origin; e.g., the size of the finite particle is the « classical radius » defined by its charge-mass ratio. In § 2, the « extended » gravitational solutions are investigated. They contain the Einstein weak gravitation theory as a special case. These forces are shown to be charge-independent. In § 3 the connection with radiation damping of the ultra-Maxwell terms in the « extended » electromagnetic solutions is investigated. Several new possibilities for avoiding the difficulties of point particles « self-force » are discussed. In § 4 some aspects of the proposed exact equations are remarked. It is shown that general relativity is properly contained in conformal relativity in the sense that every exact solution of the former gives a « pure gravitational » exact solution of the latter.

Introduction.

A previous paper (1) dealt with the kinematics of field theory admitting the conformal group in space-time as fundamental group, where the space was taken to be the full 5-dimensional one of all spheres, not just the 4-di-

(1) R. L. Ingraham: *Conformal Geometry and Elementary Particles*, in *Nuovo Cimento*, **12**, 825 (1954), hereafter referred to as I.

mensional subspace of the null spheres (events) of space-time. The consistent interpretation of that geometry implied a revision in some basic physical concepts, notably that of « elementary particle ». It suggested that the notion of a particle as a point or small rigid 3-sphere (we shall refer to these concepts as the *classical point particle or classical finite particle*, respectively, for convenience) be replaced by the concept of it as a finite complex in space-time representing the locus of events influenced (or influenceable?) by it by means of signals propagating with some finite velocity. This complex representing the particle at any instant t would be the hyperboloid of events $(\boldsymbol{\xi}, c\tau)$

$$(\boldsymbol{\xi} - \boldsymbol{r})^2 - c^2(\tau - t)^2 = \varepsilon\lambda^2 \qquad (\varepsilon = \pm 1), \tag{1}$$

namely the space-time « sphere » of center $(\boldsymbol{r}, ct)$ and radius squared (*) $\varepsilon\lambda^2$ forming one of the points of our sphere space; the particle enduring throughout a certain time would be a 1-parameter family of such hyperboloids given by specifying $(\boldsymbol{r}, t, \lambda)$ as functions of some one running variable θ. In such a particle concept the 5-foldness of the coordinates of the hyperboloid indicates an extra element of structure beside spatio-temporal position to be attributed to the source of the « disturbance ». This extra element, measured by λ can only be that of finite size — hence the name *size* (of the test particle) given to λ in I. E.g., for $\varepsilon = -1$, (1) can be interpreted as the space-time locus of a light pulse emitted at $\tau = t$ and propagating from the limb of an ordinary 3-sphere of center $\boldsymbol{r}$ and radius λ. Now according to the motion equations (which give $(\boldsymbol{r}, t, \lambda)$ as functions of θ) in the force-free case (2), the law of growth of λ, the size of the source, is such that this source must be interpreted as a « sphere of influence » attached rigidly to a classical particle and growing with a velocity which rapidly approaches c, rather than as a classical particle itself. The picture of the new particle which emerges is then something like the following: a rapidly expanding « influence sphere » moving rigidly with a classical particle which at every instant by a sort of Huyghen's Principle is the source of disturbance (1).

In the free particle paths (2), beside the parameters familiar from point particle motion, occurs a length λ_0. This length turns out to be the lower limit of the size λ of the influence sphere. I.e., the influence sphere starts growing *not* from a point but from the periphery of a 3-sphere of this radius. In this sense we associate a classical finite particle of radius

$$r_{cl} = \lambda_0 \tag{2}$$

(2) Cf. I, (3·2) or § 1 of this paper. N.B. the λ_0 of (2) equals the $\lambda_0 \, |1 - v^2/c^2|^{-\frac{1}{2}}$ of I. Similarly the l_θ of (3) equals the $l_0 |1 - u^2/c^2|^{\frac{1}{2}}$ of I.

(3) Cf. I, (2·3).

(*) *Added in proof*: $\varepsilon = +1$ is excluded as unphysical. Hence ε should be read as -1 in this paper.

with our new particle. The introduction into the theory of an associated classical finite particle in this way is accompanied by no loss of covariance of course. It remains for the further working out of the theory to decide whether this is an apt definition (whether for example certain integrals of physically interesting densities which diverge in the classical theory result finite in this theory because $r_{cl} \neq 0$). All this in position space. Due to the automatic p-q reciprocity of this theory ([3]), all the above considerations admit an immediate translation into the momentum space language. Thus one extra parameter ([4]) $l_0 \sim L$ beyond the familiar ones appears in the momentum space free motion equations ([2]). This determines a lower limitto the proper mass, that is, a *residual proper mass*

$$m_r = \frac{\hbar}{l_0 c} \tag{3}$$

of the particle present even at zero energy. This furnishes a correction term to the Einstein mass-energy relation for classical point particles ([5]) and represents a generalization of it to the case of a classical finite particle. The non-vanishing residual mass m_r is to be thought of as the manifestation in p-space, corresponding to the appearance of the non-vanishing size r_{cl} in q-space, of the finiteness of the associated classical particle.

One of the main tasks of this paper is to show the electromagnetic origin of these finite effects. This requires going from the force-free to the weak field case. The proposed field and particle motion equations for this case are given in § 1. One gets all the solutions of the classical Maxwell theory in vacuo as special cases. When the motion under these « classical electromagnetic » solutions is compared with the Lorentz law of motion, it can then be inferred that

$$r_{cl} = \left(\frac{e}{m}\right)_p \frac{e}{c^2}, \tag{4}$$

where $(e/m)_p$ is the absolute value of the charge-mass ratio of the particle P, $-e$ is the electron charge ([6]). Thus, for P an electron, r_{cl} is just the « classical

([4]) Read the symbol $\sim$ « of physical dimensions ».

([5]) Here it is useful to recall that EINSTEIN originally derived this relation for *point* particles, and it was only subsequently extended by pure induction to general material configurations. Moreover, the relation was obtained by requiring the particle motion equations in p-space to be (Lorentz) covariant, which is precisely analogous to the derivation of this generalized mass-energy relation here.

([6]) Note that the test particle charge occurs only in the combination $(e/m)_p$, hence (4) rather than $e_P^2/m_P c^2$ appears.

radius », that distance making the lower limit of the validity of the classical Maxwell theory. The residual mass m_r turns out to depend not only on $(e/m)_p$ but also on other parameters of the path. It is also shown that in « classical electromagnetic » motion just as in free motion r_{cl} is the lower limit of λ.

The « extended » electromagnetic force tensor ([7]) $f_{\alpha\beta}$ ($\alpha, \beta = 1, ..., 5$) which appears in this theory is a 5×5 quantity which is in general a function of all five coordinates $(\boldsymbol{r}, t, \lambda)$. For the « classical electromagnetic » solutions, $f_{5m} = 0$ and ([7]) f_{mn} ($m, n = 1, ..., 4$) are independent of λ. A comparison of the equations of § 1 with the classical field and motion equations shows (§ 3) that the extra components f_{5m} would account, in the classical picture, for non-vanishing current and self-force terms, resp.. A sample calculation of particle motion under an extended electromagnetic solution whose classical part corresponds to a uniform constant magnetic field is carried out in § 3 to ascertain whether the effect of radiation damping can be produced in this way without recourse to the notion of particle self-force with its well known attendant difficulties. A damped motion in fact results, where, however, the asymptotic motion is inverse-quadratically rather than exponentially damped, due to the appearance of elliptic functions in the solution. It is shown that one effect of non-vanishing extended electromagnetic forces ($f_{5m} \not\equiv 0$) is to decrease the lower limit of λ: $0 < glb\, \lambda < r_{cl}$. In § 3 are also given arguments for defining λ/c as the proper time of the associated classical finite particle (2), and some remarks on how this covariant introduction of finite particles might modify difficulties heretofore encountered in the « self force » notion.

The field equations also describe a set of potentials ([7]) $k_{\alpha\beta}$ called « extended » gravitational potentials because one gets all the solutions of the Einstein weak field gravitation theory as special cases (§ 2). The equations satisfied by the extra potentials k_{5m} assuming $k_{55} = 0$ in the absence of ordinary gravitation and extended electromagnetic forces as well as the motion equations under these forces are written out to give a rough idea of their nature. An important result is that the $k_{\alpha\beta}$-forces are charge-independent.

In § 4 with the giving of the proposed exact (non-linear) field and motion equations, of which those of § 2 are linearizations, it can be demonstrated that the extended gravitational forces are rigorously charge-independent. The relation of general relativity to this theory is shown to be the following: general relativity is properly contained in this theory in the sense that every exact solution of the former gives at once a « pure gravitational » exact solution of the latter.

([7]) General alphabet convention: greek letters α, β, γ, etc., go from 1 to 5; roman letters l, m, n, etc., from 1 to 4.

1. – The Weak Field Equations and the Classical Radius.

According to Conformal Relativity, the paths of free particles (8) are given by

$$(1.1) \qquad \boldsymbol{r} - \boldsymbol{r}_0 = \boldsymbol{v}(t - t_0), \qquad \lambda = |1 - v^2/c^2|^{\frac{1}{2}} \left\{ c^2(t - t_0)^2 + \frac{\lambda_0^2}{|1 - v^2/c^2|} \right\}^{\frac{1}{2}}$$

in position space, and by

$$(1.2) \qquad \boldsymbol{p} - \boldsymbol{p}_0 = \boldsymbol{u}\left(\frac{E - E_0}{c}\right), \quad mc = |1 - u^2/c^2|^{\frac{1}{2}} \left\{ \left(\frac{E - E_0}{c}\right)^2 + \frac{(\hbar/l_0)^2}{|1 - u^2/c^2|} \right\}^{\frac{1}{2}}$$

in momentum space. $(\boldsymbol{r}, t)$, $(\boldsymbol{p}, E)$ are position and time, momentum and energy resp. relative to a Lorentz frame; m is proper mass; λ/c is proper time (see § 3 below ») and will also be called the *size* (« of the particle » or « of the source). The integration constants $\boldsymbol{r}_0$, t_0, $\boldsymbol{v}$, λ_0 and $\boldsymbol{p}_0$, E_0, $\boldsymbol{u}$, l_0 have the dimensions suggested by the notation. In addition to these parameters there is one more, $\varepsilon = \pm 1$, occurring in the conditions (9) $\operatorname{Sgn}(1 - v^2/c^2) = \operatorname{Sgn}(1 - u^2/c^2) = -\varepsilon$ for $u, v \neq c$. Eq. (1.2) can be obtained from eq. (1.1), in virtue of the automatic p-q reciprocity which characterizes all the equations of this theory, by making the direct substitution (7) $x^\alpha \to p^\alpha$ in (1.1),

$$(1.3) \qquad x^1 = x, \text{ etc.}, \quad x^4 = ct, \quad x^5 = \lambda; \qquad p^1 = p_x, \text{ etc.}, \quad p^4 = \frac{E}{c}, \quad p^5 = mc,$$

and by renaming the constants (in virtue of their new dimensions). The parameters $\boldsymbol{r}_0$, t_0, $\boldsymbol{v}$, $\boldsymbol{p}_0$, E_0, and $\boldsymbol{u}$ play roles familiar from point particle mechanics. The further parameters λ_0 and l_0 account for the non-zero size and non-zero residual mass of the associated finite classical particle (cf. Introduction). We shall show that these effects are in fact of electromagnetic origin: that the length λ_0 characterizes simply the charge-mass ratio of the particle whereas the length l_0 is related to this ratio in a more complicated way, depending also on $\boldsymbol{r}_0$, t_0, and $\boldsymbol{v}$. Thus although the charge in the force free case cannot of course manifest itself by an electromagnetic deflection of the particle from uniform motion, nevertheless it shows up in prescribing the finiteness in position and momentum space of the associated classical particle.

To do this, we first write down the field and motion equations adopted

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(8) In this paper we restrict the discussion to the « non-null » particles, i.e., those along whose paths the angle element $\mathrm{d}\theta \neq 0$.

(9) The absolute value signs in (1.1), (1.2) are thus not superfluous. $\varepsilon = +1$ gives space-like motion. (See the note added in proof.). By the exponent $\frac{1}{2}$ we shall always mean the positive square root.

for the case of weak fields. The linearized field equations ([10]) run

$$\text{(1.4)} \qquad \begin{cases} \text{a)} \quad H_{\alpha\beta} + \dfrac{3}{\lambda} M_{\alpha\beta} = 0 \\ \text{b)} \quad \partial_\alpha(\lambda^{-1} f^{\alpha\beta}) = 0 \,; \quad \partial_{\langle\alpha} f_{\beta\gamma\rangle} = 0 \end{cases}$$

with

$$\text{(1.5)} \qquad \begin{cases} H_{\alpha\beta} \equiv \dfrac{1}{2}\nabla^2\mu_{\alpha\beta} - \dfrac{1}{2}(\partial_\alpha\tau_\beta + \partial_\beta\tau_\alpha) + \dfrac{1}{2}\delta^\varepsilon{}_{\alpha\beta}\partial_\gamma\tau^\gamma\,, \qquad \tau_\alpha \equiv \partial_\beta\mu^\beta{}_\alpha \\ M_{\alpha\beta} \equiv -\dfrac{\varepsilon}{2}[\partial_\alpha\mu_{\beta 5} + \partial_\beta\mu_{\alpha 5} - \partial_5\mu_{\alpha\beta} - \\ \qquad\qquad - \dfrac{1}{3}(\delta^\varepsilon{}_{\alpha 5}\partial_\beta\mu + \delta^\varepsilon{}_{\beta 5}\partial_\alpha\mu - \delta^\varepsilon{}_{\alpha\beta}\partial_5\mu) - 2\delta^\varepsilon{}_{\alpha\beta}\tau_5]\,, \end{cases}$$

where the fields to be solved for, $\mu_{\alpha\beta}$ (symmetric ([4]), ~ 1) and $f_{\alpha\beta}$ (skew ([4]), $\sim L^{-2}$) are assumed small of the same order. Here $\partial_\alpha = \partial/\partial x^\alpha$ where x^α are given in terms of the physical variables by (1.3), the bracket $\langle\ \rangle$ around the indices in (1.4) b) means that the cyclical sum is to be taken; $\delta^\varepsilon{}_{\alpha\beta} = (1, 1, 1, -1, -\varepsilon)$ on the diagonal and zero off it; indices are raised with $\delta^{\varepsilon\,\alpha\beta}$; $\nabla^2 \equiv \delta^{\varepsilon\,\alpha\beta}\partial^2{}_{\alpha\beta}$ is the Laplacian with respect to the metric $\delta^\varepsilon{}_{\alpha\beta}$; $\mu \equiv \delta^{\varepsilon\,\alpha\beta}\mu_{\alpha\beta}$.

The base space here is the 5-dimensional one of all « spheres » (1) in Minkowski space; the fundamental differential invariant $\mathrm{d}\theta$, the angle under which two neighboring spheres intersect. In *natural* sphere coordinates ξ^α (ξ^m=orthonormal cartesian coordinates of the center, ξ^5= absolute value of the real or pure imaginary radius R, $\varepsilon = \mathrm{Sgn}\, R^2$) this infinitesimal angle is given by

$$\text{(1.6)} \qquad \mathrm{d}\theta^2 = \varepsilon(\xi^5)^{-2}\big((\mathrm{d}\xi)^2 - \varepsilon(\mathrm{d}\xi^5)^2\big)$$

where ([11]) $(\mathrm{d}\xi)^2 \equiv g_{mn}\mathrm{d}\xi^m\mathrm{d}\xi^n$, $g_{11} = g_{22} = g_{33} = -g_{44} = 1$, $g_{mn} = 0$, $(m \neq n)$. Both x^α and p^α are natural coordinates. The linearized motion equations ([8,10]) in terms of θ as running variable are

$$\text{(1.7)} \qquad \frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\theta^2} - \delta^\alpha_5\lambda - 2\,\frac{\mathrm{d}\log\lambda}{\mathrm{d}\theta}\,\frac{\mathrm{d}x^\alpha}{\mathrm{d}\theta} + \\ + \left(\partial_\beta k^\alpha{}_\gamma - \frac{1}{2}\,\partial^\alpha k_{\beta\gamma}\right)\frac{\mathrm{d}x^\beta}{\mathrm{d}\theta}\,\frac{\mathrm{d}x^\gamma}{\mathrm{d}\theta} - \varepsilon\delta^\alpha_5\lambda^{-1}k_{\beta\gamma}\,\frac{\mathrm{d}x^\beta}{\mathrm{d}\theta}\,\frac{\mathrm{d}x^\gamma}{\mathrm{d}\theta} + \beta\varepsilon\lambda^2 f^\alpha{}_\beta\,\frac{\mathrm{d}x^\beta}{\mathrm{d}\theta} = 0\,.$$

([10]) Hereafter we shall write down equations only in their position space form. By the q-p reciprocity the momentum space equations can be immediately obtained by replacing x^α by p^α in them and renaming the various tensors to account for their new dimensions.

([11]) The notation $(\ )^2$ will hereafter always mean the square of the MINKOWSKI length of the first four components of a 5-component quantity.

Here $k_{\alpha\beta} \equiv \mu_{\alpha\beta} - \frac{1}{3}\delta^{\varepsilon}{}_{\alpha\beta}\mu$; indices are raised with $\delta^{\varepsilon\alpha\beta}$; and β is a certain constant [12] independent of the path left undetermined for the moment. The force-free paths, integral curves of these equations with $k_{\alpha\beta} = f_{\alpha\beta} = 0$, are non-null geodesics of the Riemannian space of constant curvature $+1$ with metric (1.6). If now we substitute the running variable s for θ, where ds is defined by

$$\frac{ds}{d\theta} = \varkappa\lambda^2 , \tag{1.8}$$

$\varkappa$ a constant $\sim L^{-1}$ depending on the path, equations (1.7) in the forcefree case ($k_{\alpha\beta} = f_{\alpha\beta} = 0$) simplify to

$$\frac{d^2x^m}{ds^2} = 0 , \qquad \frac{d^2\lambda}{ds^2} - \varkappa^{-2}\lambda^{-3} = 0 . \tag{1.9}$$

From the first set of equations [11], $|(dx)^2| = \text{const.} \times ds^2$, and we now choose $\varkappa$ to make this proportionality constant equal to unity, i.e., s/c the proper time τ of the associated *point* particle (by this we shall always mean the center x^m of our finite particle). Then the integral curves are

$$x^m = a^m s + b^m , \qquad \lambda^2 = s^2 + \varkappa^{-2} \qquad [(a)^2 = \varepsilon] \tag{1.10}$$

where the identity

$$\left(\frac{dx}{ds}\right)^2 = \varepsilon\left[\left(\frac{d\lambda}{ds}\right)^2 + \varkappa^{-2}\lambda^{-2}\right] , \tag{1.11}$$

following from (1.6) and (1.8), has eliminated one constant, and we have made the s-origin correspond to minimum λ. These are identical with (1.1) where $\boldsymbol{a} = |1 - v^2/c^2|^{-\frac{1}{2}}\boldsymbol{v}/c$, $a^4 = |1 - v^2/c^2|^{-\frac{1}{2}}$, $\boldsymbol{b} = \boldsymbol{r}_0$, $b^4 = ct_0$, and

$$|\varkappa^{-1}| = \lambda_0 \equiv r_{cl} . \tag{1.12}$$

Hence it is the connection with the charge-mass ratio of the length $\varkappa^{-1}$ appearing in (1.8) so normalized that s has the interpretation given there that we wish to establish. By this substitution (1.8), the motion equations (1.7)

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[12] We shall take $\beta > 0$ without loss of generality, since the field equations are invariant under $f_{\alpha\beta} \to -f_{\alpha\beta}$.

in the general case become

$$(1.7') \quad \frac{d^2x^\alpha}{ds^2} - \delta_5^\alpha \varkappa^{-2}\lambda^{-3} + \\ + \left(\partial_\beta k^\alpha_{\ \gamma} - \frac{1}{2}\,\partial^\alpha k_{\beta\gamma}\right)\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} - \delta_5^\alpha \lambda^{-1} k_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} + \\ + \varepsilon\varkappa^{-1}\beta f^\alpha_{\ \beta}\frac{dx^\beta}{ds} = 0\,.$$

Consider a solution of (1.4) in which only the $f_{mn} \not\equiv 0$. Then the set a) are satisfied identically, and the set b) reduce to

$$\partial_m f^m_{\ n} = 0\,, \quad \partial_{\langle m} f_{pq\rangle} = 0\,, \quad \partial_5 f_{mn} = 0\,.$$

Such solutions exist; we shall call them *classical electromagnetic* solutions, for every solution $F_{mn}(x^p)$ of Maxwell's equations in vacuo furnishes a solution f_{mn} which can be written $C\beta^{-1}e^{-1}F_{mn}$ of (1.8). Here $-e$ is electron charge and $C \sim 1$, a constant. C can be freely assigned in this linear theory with its lack of coupling between the $k_{\alpha\beta}$ and $f_{\alpha\beta}$-fields, and we shall choose it to be unity for simplicity ([13]). Putting this solution into (1.7)', we get

$$(1.14) \quad \frac{d^2x^m}{ds^2} = -\,\varepsilon\varkappa^{-1}e^{-1}F^m_{\ n}\frac{dx^n}{ds}\,, \qquad \frac{d^2\lambda}{ds^2} - \varkappa^{-2}\lambda^{-3} = 0\,.$$

Multiplying the first set by dx_m/ds, by the antisymmetry of F_{mn} we infer that $s/c = \text{const.} \times \tau$; hence s/c is still the proper time of the associated point particle since letting $F_{mn} \to 0$, we infer by continuity that this constant must be unity. So we substitute $s = c\tau$ in (1.14) and compare with the Lorentz law of motion for the charged point particle P at x^m of charge-mass ratio $(e/m)_P > 0$ and sign of charge Sgn e_P in the field F_{mn}:

$$(1.15) \quad \frac{d^2x^m}{d\tau^2} = \text{Sgn}\, e_P \left(\frac{e}{m}\right)_P \frac{1}{c}\,F^m_{\ n}\frac{dx^n}{d\tau}\,.$$

We infer

$$(1.16) \quad |\varkappa^{-1}| = \left(\frac{e}{m}\right)_P \frac{e}{c^2}\,, \qquad \text{Sgn}\,\varkappa^{-1} = \text{Sgn}\, e_P\,,$$

([13]) We expect, on the basis of the non-linear theory of § 4, that C will not be arbitrary, but fixed in terms of the known connection of the $k_{\alpha\beta}$-forces with gravitation, etc.. Hence strictly we can infer only numerical proportionality rather than equality in (1.16) below.

which proves the assertion (4). We see also that the two possible signs of $\varkappa^{-1}$ (ds assumed >0) correspond to the two possible signs of the charge for non-null particles.

The second equation (1.14) gives λ of the form (1.10) just as in the force-free case. Hence under classical electromagnetic forces, the behavior of the particle size is no different from its force-free behavior — in particular, its lower limit is still r_{el}.

The identification of the residual mass (3) in terms of familiar physical parameters by the same method fails because the momentum-space analogue of (1.15) is lacking in the classical Maxwell theory. We can however make use of the inversion

$$p^\alpha = -\hbar x^\alpha/[x]^2, \quad [x]^2 \equiv (x)^2 - \varepsilon(x^5)^2, \tag{1.17}$$

the particular conformal sphere transformation tentatively supposed in I to connect the (equivalent) q- and p-frames. This transformation takes the q-paths (1.1) into the p-paths ([14]) (1.2) and gives the various p-parameters as functions of the q-parameters. This can be most easily worked out using the formulas given in I, App. III. m_r depends not only on the « electromagnetic » radius $\varkappa^{-1}$ but also on the other q-parameters. This means physically that, *using always the same p-frame* (1.12), the « same » particle (meaning for fixed Sgn e_p and $(e/m)_p$, say) would be observed to have different residual masses depending on the particular trajectory in space-time.

2. – The Charge-Independence of the Extended Gravitational Forces.

By (1.7′) motion under a solution $k_{mn}(x^p)$, $k_{5\alpha} = 0$, $f_{\alpha\beta} = 0$ of (1.4) has just the form in the first four components of point particle motion in the linearized Einstein gravitation theory ([15]), where k_{mn} plays the role of the small discrepancies of the Riemannian metric from the Lorentz metric δ_{mn}. Substituting $\mu_{\alpha\beta} = k_{\alpha\beta} - \frac{1}{2}\delta^\varepsilon_{\alpha\beta}k$, $k \equiv \delta^{\varepsilon\alpha\beta}k_{\alpha\beta}$ in (1.4) it is readily verified that the k_{mn} satisfy just the field equations of linearized general relativity. (In fact we shall prove in § **4** a much stronger result, namely that general relativity itself is properly contained in conformal relativity.) Now if we call the solutions $\{k_{\alpha\beta}, f_{\alpha\beta}\}$ of (1.4) for which all components having an index 5 vanish, and the rest are independent of λ *classical* solutions, we have the result from

([14]) Proof: the motion equations are form-invariant under the whole group of conformal sphere transformations, in particular (1.17).

([15]) P. G. BERGMANN: *An Introduction to the Theory of Relativity* (New York, 1942), pp. 180-184.

this and § 1 that any classical solution describes at most a superposition of Maxwell electromagnetism and Einstein gravitation.

By (1.7′) the constant $\varkappa^{-1}$ does not occur as a coefficient of the $k_{\alpha\beta}$-terms and hence in view of (1.16) *these forces are charge-independent.* Thus not only gravitation but also the forces due to the potentials $k_{5\alpha}$ share this property. The fundamental reason for this may be seen in the fact that the $k_{\alpha\beta}$ terms in (1.7) are quadratic in the velocities, whereas the $f_{\alpha\beta}$ term is linear in them. Thus when the substitution (1.8) is made in (1.7) and the acceleration term cleared of coefficients, the charge-dependent constant cancels out of the $k_{\alpha\beta}$-terms but leaves a factor $\varkappa^{-1} \propto (e/m)_p$ in the $f_{\alpha\beta}$ term in (1.7′). This charge-independence of the *extended gravitational* forces is rigorous, as will be shown in § 4.

For a solution in which only the $k_{5m} \equiv \psi_m$ do not vanish say, the field equation (1.4) reduce to (App. I)

$$(2.1)\qquad \begin{cases} \text{a)} & \partial_5(\partial_m\psi_n + \partial_n\psi_m) - 3\lambda^{-1}(\partial_m\psi_n + \partial_n\psi_m) = 0 \\ \text{b)} & \Box^2\psi_m = 0\,, \\ \text{c)} & \partial_m\psi^m = 0\,. \end{cases}$$

From the set a) we get

$$(2.2)\qquad \partial_m\psi_n + \partial_n\psi_m = q_{mn}(x^p)\lambda^3\,, \qquad \partial_m q^m{}_n = 0\,, \qquad q^m{}_m = 0\,,$$

the last two equations following from b) and c). It is remarkable that the « mass term » $-\varepsilon\partial_5^2\psi_m$ cancels out of b). Thus wave-like solutions are propagated with light velocity. Particular solutions of (2.1) are, for example

$$\psi_m = A_m\, e^{ik_p x^p}\lambda^3\,; \qquad A_m,\ k_m \text{ constants},\ A_p k^p = k_p k^p = 0$$

$$\psi_i = 0 \quad (i = 1, 2, 3)\,, \quad \psi_4 = A/r\lambda^3\,; \qquad A \text{ const.},\ r^2 = x^2 + y^2 + z^2\,.$$

The motion equations (1.7′) become

$$(2.3)\qquad \begin{cases} \dfrac{\mathrm{d}^2x^m}{\mathrm{d}s^2} = \dfrac{\mathrm{d}\lambda}{\mathrm{d}s}\,\Psi^m{}_n\,\dfrac{\mathrm{d}x^n}{\mathrm{d}s} - \partial_5\psi^m\left(\dfrac{\mathrm{d}\lambda}{\mathrm{d}s}\right)^2 \\ \dfrac{\mathrm{d}^2\lambda}{\mathrm{d}s^2} = \varkappa^{-2}\lambda^{-3} - \varepsilon\partial_p\psi_q\,\dfrac{\mathrm{d}x^p}{\mathrm{d}s}\,\dfrac{\mathrm{d}x^q}{\mathrm{d}s} - 2\varepsilon\lambda^{-1}\psi_p\,\dfrac{\mathrm{d}x^p}{\mathrm{d}s}\,\dfrac{\mathrm{d}\lambda}{\mathrm{d}s}\,, \end{cases}$$

where $\Psi_{mn} \equiv \partial_m\psi_n - \partial_n\psi_m$. Note that the first set, apart from the last term on the right, has electromagnetic form in which the role of the constant $(e/m)_p 1/c^2$ is taken over by the variable $\mathrm{d}\lambda/\mathrm{d}s$, whose behavior is governed by the last equation. These forces then tend to produce large accelerations in regions where $\mathrm{d}\lambda/\mathrm{d}s$ is large, that is, where the proper time λ/c of the asso-

ciated finite particle (see § 3) is appreciably different from the quantity s/c. s/c, as we have seen, is sometimes the proper time of the associated *point* particle; it is so here if in particular $\partial_5 \psi_m = 0$ (cf. the first set of (2.3)). In this case, $d\lambda/ds$ can be large only for $\lambda \lessapprox r_{cl}$ (cf. § 3).

If the $k_{5\alpha}$ terms were to describe nuclear forces, the equations (2.1) and (2.3) could be without interest for the reason that the weak field assumption on which they are based might be violated in the physical domains of interest. Another difficulty is that ψ_m may never occur unaccompanied by the other forces which we have set equal to zero. The mathematical root of this trouble is that the linearizations of a set of non-linear laws admit solutions which are the weak limits of no solutions of the non-linear laws.

3. – The Extended Electromagnetic Solutions; Radiation Damping.

Consider the classical electromagnetic solution $F_{12} = H > 0$, other $F_{mn} = 0$ corresponding to a uniform constant magnetic field H along the z-axis. One gets the particular solution for the motion

$$(3.1) \qquad x = R\cos ks\,, \quad y = R\sin ks\,, \quad z = 0 \quad ct = s\left|1 - v_0^2/c^2\right|^{-\frac{1}{2}}, \quad \lambda^2 = s^2 + \varkappa^{-2}$$

where $k = (e/m)_p H/c^2$, v_0 is the constant orbital velocity of the point particle and R the constant radius of its circular trajectory. One knows that this solution does not correspond exactly to the physical reality because due to the non-uniform acceleration ([16]) and consequent energy radiation from the particle, the motion is radiation damped. The classical Maxwell theory takes care of this situation by adding to the external force in the motion equations a self-force term, usually simply that one of the terms occurring in the expansion of the retarded field of the point particle which has the form (3-force)

$$(\tfrac{2}{3}e_p^2/4\pi c^3)(d^2\boldsymbol{v}/dt^2)\,.$$

Then these third order differential equations are solved to get the damped motion. This self-force concept is open to two general objections. 1) The point versus finite particle dilemma: an infinite term occurs in the expansion of the self-force for a point particle, the well known electrostatic contribution to its inertial mass. Thus, strictly speaking, the question of the motion of a point particle in its self-field is meaningless. Moreover, the system: field and motion equations then contain contradictions, e.g., the motion equation allow certain

([16]) Precisely, the relativistic acceleration vector of the point particle is not constant.

solutions which violate energy conservation ([17]). If one assigns the particle a finite spatial extension, one escapes the infinities but again loses Lorentz invariance ([18]). 2) The infinite versus finite order dilemma: exact solutions of the system: field and motion equations are in general excluded in principle because this would entail keeping all of the terms in the self-force expansion, hence solving infinite order differential equations. Hence by breaking off this series at some finite order, we necessarily confine ourselves to approximate solutions. Even then, the retention of third order terms (cf. above) necessary to give any damping at all, is at variance with the well-established principle that second order equations should suffice for physics.

The electromagnetic theory proposed here opens up new possibilities for explaining the physical fact of radiation damping. This is to be accomplished essentially by means of the extra degree of freedom here over the classical theory, which furnishes an extra term in both the field and motion equations. Namely, dropping the restriction to classical electromagnetic solutions, the field and motion equations (1.4) and (1.7′) in the case of no extended gravitation, $k_{\alpha\beta} = 0$, become

$$(3.2)\qquad \begin{cases} \text{a)} & \partial_m F^m{}_n = -\varepsilon(\partial_5 - \lambda^{-1})F_{5n} \\ \text{b)} & \partial_m F^m{}_5 = 0 \\ \text{c)} & \partial_{\langle m} F_{np\rangle} = 0 \\ \text{d)} & \partial_m F_{5n} - \partial_n F_{5m} = -\partial_5 F_{mn} \end{cases}$$

and

$$(3.3)\qquad \begin{cases} \dfrac{\mathrm{d}^2 x^m}{\mathrm{d}s^2} = -\varepsilon\varkappa^{-1}e^{-1}F^m{}_n \dfrac{\mathrm{d}x^n}{\mathrm{d}s} - \varepsilon\varkappa^{-1}e^{-1}F^m{}_5 \dfrac{\mathrm{d}\lambda}{\mathrm{d}s} \\ \dfrac{\mathrm{d}^2\lambda}{\mathrm{d}s^2} = \varkappa^{-2}\lambda^{-3} - \varepsilon\varkappa^{-1}e^{-1}F^5{}_m \dfrac{\mathrm{d}x^m}{\mathrm{d}s}\,, \end{cases}$$

with $F_{\alpha\beta} \equiv \beta e f_{\alpha\beta}$. In (3.2), $-\varepsilon(\partial_5 - \lambda^{-1})F_{5n}$ appears in the role of a classical 4-current s_n, whereas by (3.3) F_{5n} would account for the classical self-force term. Two avenues of investigation suggest themselves: 1) that we retain the self-force notion, that is, look for solutions for the total field $F_{\alpha\beta}$ containing a classical part « due » to the particle — meaning analogous to the Liénard-Wiechert solutions for a point particle in motion — but where now the *covariant* finiteness of the associated classical particle might be expected to rule out infinities;

([17]) Cf. say LANDAU and LIFSHITZ: *The Classical Theory of Fields* (Cambridge, 1951), Chap. 9, § 9.

([18]) Cf. say W. PAULI: *Die Allgemeinen Prinzipien der Wellenmechanik* (Ann Arbor, 1947), p. 271.

or 2) that we throw out the self-force notion entirely, i.e., see whether the damping might not result from a non-vanishing « background » F_{5n} not « due » to the particle at all. From the second viewpoint we would ascribe no field to the test particles (1.7′); theirs would be the purely passive role of suffering forces $F_{\alpha\beta}$, which would then automatically always be the « external » field. Here we shall follow up only suggestion 2 and show how a plausible extension of the solution (3.1) to the case of a background $F_{5n} \not\equiv 0$ may be obtained which exhibits a radiation damping.

As before, take $F_{12} \equiv H$, other $F_{mn} = 0$. Then by (3.2) a), $F_{5n} = \lambda\psi_n(x^p)$ where $\psi_n \sim eL^{-3}$ so that ψ_n has the dimensions of a current density vector. By d) and b), $\psi_q = \partial_q\psi$ and $\Box^2\psi = 0$, $\Box^2$ the d'Alembertian. The physical boundary conditions impose a perfect spatial homogeneity and isotropy in the x-y plane (as well as z-independence of course). The only solution which obeys these conditions is

$$\psi = \sigma_0 x^4 , \tag{3.4}$$

where σ_0 is a constant which we shall take positive. Therefore $F_{54} = \lambda\sigma_0$, other $F_{5m} = 0$. Putting this solution into (3.3) we get

$$\begin{cases} \ddot{x} = -\varepsilon k\dot{y} , & \ddot{y} = \varepsilon k\dot{x} , & \ddot{z} = 0 \\ \ddot{x}^4 = -\varepsilon k\gamma\lambda\dot{\lambda} , & \ddot{\lambda} = \varkappa^{-2}\lambda^{-3} + k\gamma\lambda\dot{x}^4 , \end{cases} \tag{3.5}$$

where k has been defined in (3.1), $\gamma \equiv \sigma_0/H$, and the dot means $\mathrm{d}/\mathrm{d}s$. x, y, z are again given by (3.1). The fourth equation has the first integral

$$\dot{x}^4 = |1 - v_0^2/c^2|^{-\frac{1}{2}} - \varepsilon k\gamma\lambda^2/2 , \tag{3.6}$$

where v_0 is the undamped ($\gamma = 0$) constant orbital velocity. Putting this into the last equation (3.5), we must solve

$$\ddot{\lambda} = \varkappa^{-2}\lambda^{-3} + k\gamma|1 - v_0^2/c^2|^{-\frac{1}{2}}\lambda - \varepsilon k^2\gamma^2\lambda^3/2 . \tag{3.7}$$

Making the substitution $p = \dot{\lambda}$. $\ddot{\lambda} = p\,\mathrm{d}p/\mathrm{d}\lambda$, this gives the integral

$$\dot{\Lambda}^2 = F(\Lambda) \equiv -\varepsilon\alpha^4\Lambda^3 + 4\alpha^2|1 - v_0^2/c^2|^{-\frac{1}{2}}\Lambda^2 + 4\Lambda - 4\varkappa^{-2} , \tag{3.8}$$

where $\Lambda \equiv \lambda^2$ and $\alpha \equiv (k\gamma)^{\frac{1}{2}}$. Hence the solution is

$$\int \frac{\mathrm{d}\Lambda}{[F(\Lambda)]^{\frac{1}{2}}} = s + \text{const.} \tag{3.9}$$

Thus Λ is an elliptic function of s. We discuss first the case $\varepsilon = -1$ (time-like motion). By the substitution $\Lambda' = \alpha^2\Lambda/4 + \frac{1}{3}|1 - v_0^2/c^2|^{-\frac{1}{2}}$, (3.9) can be thrown into the standard form

$$(3.10)\qquad \int_{\Lambda'}^{\infty} \frac{d\Lambda'}{[\Phi(\Lambda')]^{\frac{1}{2}}} = \alpha(s_\infty - s)\,; \qquad \Phi(\Lambda') \equiv 4\Lambda'^3 - g_2\Lambda' - g_3\,,$$

with

$$(3.11)\qquad \begin{cases} g_2 = \frac{4}{3}|1 - v_0^2/c^2|^{-1} - 1 \\ g_3 = -\frac{8}{9}|1 - v_0^2/c^2|^{-\frac{3}{2}} + \frac{1}{3}|1 - v_0^2/c^2|^{-\frac{1}{2}} + \frac{1}{4}(\varkappa^{-1}/\alpha^{-1})^2 \end{cases}$$

and

$$(3.12)\qquad \alpha s_\infty = \int_{\Lambda_0'}^{\infty} \frac{d\Lambda'}{[\Phi(\Lambda')]^{\frac{1}{2}}}\,,$$

where Λ_0' is the greatest real zero of $\Phi(\Lambda')$. Hence by (3.10) Λ' is simply the Weierstrass $\wp$ function of $\alpha(s_\infty - s)$ with real invariants [19] g_2, g_3. Expressing the solution once more in terms of λ^2 we get

$$(3.13)\qquad \lambda^2 = (4/\alpha^2)[\wp(\alpha(s_\infty - s);\, g_2, g_3) - \tfrac{1}{3}|1 - v_0^2/c^2|^{-\frac{1}{2}}]\,.$$

Putting this into (3.6) and integrating, one obtains for the time

$$(3.14)\qquad x^4 = \tfrac{1}{3}|1 - v_0^2/c^2|^{-\frac{1}{2}} s + 2/\alpha\zeta(\alpha(s_\infty - s);\, g_2, g_3) + \text{const.}$$

where ζ is the Weierstrass ζ function [20] with the same invariants g_2, g_3.

Thus the motion is given by the first three equations of (3.1) with (3.13) and (3.14) in terms of the parameter s. This parameter has the domain $[0, s_\infty]$, $s_\infty \equiv \Omega/\alpha$ where $\Omega < \infty$ is the real half-period of the doubly periodic $\wp$ function of invariants g_2, g_3. If $x_4/c]_{s=0}^{s=s_\infty}$ be called the *eternity* of the particle, then by (3.14) the particle has an infinite eternity since the ζ function has a simple pole at the origin. Hence in terms of Lorentz time the particle

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[19] E. WHITTAKER and H. WATSON: *Modern Analysis* (Cambridge, 4th Ed., 1940), p. 437.

[20] E. WHITTAKER and H. WATSON: op. cit., p. 445.

still runs through an infinite lifetime. By (3.13) λ^2 has the range $[\lambda_0^2, +\infty]$ because $\wp$ function has a double pole at the origin. $\lambda_0^2 \equiv \Lambda_0$, the square of the lower limit of the particle size, is computed as the greatest real zero of $F(\Lambda)$. Since by (3.8) $F|_{\Lambda=0} = -4\varkappa^{-2} < 0$, $\lim_{\Lambda \to +\infty} F = +\infty$ and $F(\Lambda)$ is monotone $[\Lambda > 0]$, we draw the important conclusion that $\lambda_0^2 > 0$. In fact from (3.8) $F(\varkappa^{-2}) > 0$, hence $0 < \lambda_0 < \varkappa^{-1}$ when $\alpha \neq 0$, so that one effect of the damping is to decrease the lower limit of λ below the classical radius. The angular velocity $\omega \equiv \mathrm{d}\varphi/\mathrm{d}t = kc(\dot{x}^4)^{-1}$ is given by (3.6) and (3.13), so that the orbital velocity is

$$R\omega = \frac{Rkc}{2\wp(\alpha(s_\infty - s); g_2, g_3) + \frac{1}{3}|1 - v_0^2/c^2|^{-\frac{1}{2}}}. \tag{3.15}$$

This goes to zero as $s \to s_\infty$, the asymptotic damping being quadratic due to the double pole of $\wp$ at the origin. It is interesting to see why the asymptotic damping comes out quadratic, and not, say, exponential. First of all, for $\gamma = 0$ by (3.6) there is no damping and the integral, of (3.8) is the force-free solution $\Lambda = s^2 + \varkappa^{-2}$. The presence of a non-zero γ brings into being a Λ^2 term and also a cubic terms Λ^3 in (3.8). If the cubic term were absent, (3.8) would give an exponential solution $\Lambda(s)$ which via (3.6) would lead to an exponential damping *in* s. But this would be only a linear damping in t. But the cubic term eventually prevails, hence the appearance of elliptic functions with their pole-singularities and the asymptotic quadratic damping.

For $\varepsilon = +1$, the behavior is physically anomalous (*). The leading term in (3.8) is negative for $\Lambda > 0$, hence a new physical situation arises. The parameter s has a finite domain $[0, s_1]$, $s_1 < \infty$ as before of course, but now the range of Λ is restricted to that finite segment $[\Lambda_0, \Lambda_1]$ of the positive Λ axis for which $F(\Lambda) \geqslant 0$. Λ comes out to be a linear fractional transform of a Weierstrass $\wp$ function in s. It can easily be shown by (3.6) that x^4 then has a finite range too, i.e., that the particle has a finite eternity. By (3.6) the orbital velocity does not decrease but increases monotonically. If the upper limit Λ_1, obeys

$$\Lambda_1 \geq \frac{2}{k\gamma|1 - v_0^2/c^2|^{\frac{1}{2}}},$$

then it attains $+\infty$ at a finite time, and for the strict inequality changes sense at this time and increases from $-\infty$. At this same critical time the sense of time reverses by (3.6). This unphysical behavior is presumably connected with the space-like motion holding for $\varepsilon = +1$.

(*) See note added in proof in the Introduction.

The identity (1.11) can be rewritten

$$\left(\frac{d\lambda}{ds}\right)^2 - \left|\left(\frac{dx}{ds}\right)^2\right| = \left(\frac{\varkappa^{-1}}{\lambda}\right)^2 . \tag{3.16}$$

Hence λ/c admits interpretation as the proper time of the associated classical finite particle of radius $r_{cl} = |\varkappa^{-1}|$. For $|(dx)^2|^{\frac{1}{2}}/c$ is the proper time of the associated point particle; and λ/c, which differs essentially from this only at influence-sphere sizes of the same order as the size of the finite particle ($\lambda \cong |\varkappa^{-1}|$), goes over into it for $\lambda \gg |\varkappa^{-1}|$, where the finite particle relative to the influence sphere may be treated as point-like. We see by the motion (1.7′) for $\alpha = 5$ that there is a strong (inverse cube) repulsive force keeping λ away from zero. Hence we may expect that for the motion under any of the forces described in this theory the lower limit of λ will be greater than zero. This has in fact been verified in all the cases studied up to now.

4. – Remarks on the Exact Equations.

The exact, non-linear field equations proposed are ([21])

a) $$G_{\alpha\beta} + \tfrac{1}{2}T_{\alpha\beta} - 6\gamma_{\alpha\beta} = 0$$

b) $$1/|\gamma|^{\frac{1}{2}}\partial_\beta(|\gamma|^{\frac{1}{2}}f^{\beta\alpha}) = 0$$

c) $$\partial_{\langle\alpha}f_{\beta\gamma\rangle} = 0$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}R$, $R = R_{\delta\varepsilon}\gamma^{\delta\varepsilon}$, $R_{\alpha\beta}$ is the Ricci tensor of $\gamma_{\alpha\beta}(x^{\gamma})$, the general angle metric with respect to general sphere coordinates x^{γ}, $\gamma \equiv \mathrm{Det}\,\gamma_{\alpha\beta}$, indices are raised with $\gamma^{\alpha\beta}$, and

$$T_{\alpha\beta} = f'_{\alpha}f_{\gamma\beta} - \tfrac{1}{4}\gamma_{\alpha\beta}f^{\gamma\delta}f_{\gamma\delta} . \tag{4.2}$$

E.g., the field free case: $f_{\alpha\beta} = 0$ and $\gamma_{\alpha\beta}$ given by (1.6) in natural coordinates is a solution of these equations. The weak field equations (1.4) are a linear-

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([21]) Cf. R. INGRAHAM: *Conformal Relativity*, in *Nuovo Cimento*, 9, 886 (1952), p. 896. The first set should be corrected to read as in (4.1) a. (The present $f_{\alpha\beta}$ is twice the $f_{\alpha\beta}$ used there.) The extra term $-6\gamma_{\alpha\beta}$ is in fact necessary to make the force-free case a solution of these equations.

ization of (4.1) (see App. II). The exact motion equations run ([8,10])

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\theta^2}+\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}\frac{\mathrm{d}x^\beta}{\mathrm{d}\theta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\theta}+\beta f^\alpha{}_\beta\frac{\mathrm{d}x^\beta}{\mathrm{d}\theta}=0\,, \tag{4.3}$$

β the same pure number constant appearing in (1.7). $\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}$ are the Christoffel symbols of $\gamma_{\alpha\beta}$. (4.3) with $f_{\alpha\beta}$ set equal to zero are simply the non-null geodesics of our curvilinear sphere space. The paths (4.3) are characteristic of the particular geometry defined by this sphere space with angle metric $\gamma_{\alpha\beta}$ and superimposed bivector $f_{\alpha\beta}$ obeying (4.1) in the sense that they can be derived from these field equations by a definite mathematical procedure involving taking the limit of a certain infinite sequence of such spaces ([22]). There is some hope of uniquely fixing the number β this way. We mention this to emphasize that the addition of the invariant electromagnetic term to the geodesic part in (4.3) is not an ad hoc device geometrically, for it is these paths rather than the geodesics of $\gamma_{\alpha\beta}$ which are special for the geometry defined by (4.1). The weak field motion equations (1.7) are the linearization of (4.3).

If in a particular coordinate system, we make the substitution

$$g_{\alpha\beta}\equiv\varepsilon(x^5)^2\gamma_{\alpha\beta} \tag{4.4}$$

and at the same time introduce the parameter s in (4.3) via (1.8), we obtain

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2}+\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}_g\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\frac{\mathrm{d}x^\gamma}{\mathrm{d}s}-\delta^\alpha_5\varkappa^{-2}(x^5)^{-3}+\varepsilon\varkappa^{-1}\beta f^\alpha{}_\beta\frac{\mathrm{d}x^\beta}{\mathrm{d}s}=0\,, \tag{4.5}$$

where $\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}_g$ means the Christoffel symbol of the $g_{\alpha\beta}$ and an underlined upper index means that it has been raised with $g^{\alpha\beta}$. Here we see that that $\varkappa$ cancels out of the (extended) gravitational term but leaves a factor $\varkappa^{-1}$ in the (extended) electromagnetic term for precisely the same reasons as before (cf. § 2). Hence the extended gravitational forces (represented by $g_{\alpha\beta}$, or $\gamma_{\alpha\beta}$) are rigorously charge-independent as maintained in § 2.

An important conclusion of a very fundamental nature can be drawn from the fact that all fields represented here are comprised in two sphere-geometrical tensors $\gamma_{\alpha\beta}$ and $f_{\alpha\beta}$, where the formeris the angle-metric. We maintain that the extended gravitational forces are ineradicable, but the extended electromagnetic forces are not. By that we simply mean that the $f_{\alpha\beta}$-forces cannot be brought into existence by a coordinate transformation if they vanish in some one system.

([22]) Cf. for the 4-dimensional case L. INFELD and A. SCHILD: *Rev. Mod. Phys.*, **21**, 408 (1949); D. M. CHASE (to appear).

This simply rephrases the statement that $f_{\alpha\beta}$ is a tensor. The $\gamma_{\alpha\beta}$-forces, on the other hand, are ineradicable: since $\gamma_{\alpha\beta}$ can never vanish, if any of its components vanish in some system, they can always be recreated by a suitable coordinate change. Needless to say, the same remark goes for $f_{\alpha\beta}$ if any of its components are non-zero. E.g., in the weak field-case if a classical electromagnetic solution is subjected to a uniform accelerative change of frame ([23]), non-vanishing «extended» components f_{5m} are in general created.

The relation of the exact theory proposed here to general relativity is the following. General relativity is properly contained in this theory in the following sense: every solution $g_{mn}(x^p)$ for the metric of Einstein's field equations

$$G_{mn} \equiv R_{mn} - \tfrac{1}{2} g_{mn} R = 0$$

gives the exact solution of (4.1)

$$g_{mn} = g_{mn}(x^p), \quad g_{5m} = 0, \quad g_{55} = -\varepsilon; \quad f_{\alpha\beta} = 0, \tag{4.6}$$

where $g_{\alpha\beta}$ in the left members is the quantity defined in (4.4). This is shown most easily by first making the substitution (4.4) in the field equations (4.1) a) (App. II). That this non-invariant analytical trick might be a convenient tool for investigating exact solutions can be seen from the fact that in the force-free case $\gamma_{\alpha\beta}$ in natural coordinates is still a function of x^5 (cf. (1.6)), whereas $g_{\alpha\beta}$ is constant.

For the solution (4.6) the exact motion equations (4.5) reduce to the equations of the geodesics of the Riemannian 4-space with metric g_{mn} and length element ([24]) ds plus the equation $d^2x^5/ds^2 - \varkappa^{-2}(x^5)^{-3} = 0$ which integrates to $(x^5)^2 = s^2 + \varkappa^{-2}$. Hence such exact solutions can be called *pure gravitational* solutions. Of course the above simple form is coordinate-dependent — a general coordinate transformation introduces non-zero g_{5m} and destroys the x^5-independence of $g_{\alpha\beta}$.

In particular we get the «Schwarzschild» pure gravitational solution: $f_{\alpha\beta} = 0$ and angle metric given by

$$\varepsilon(x^5)^2 \, d\theta^2 = (1 - 2m/r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) - \\ - (1 - 2m/r)(dx^4)^2 - \varepsilon(dx^5)^2, \tag{4.7}$$

m an integration constant. An exact solution we should very much like to

([23]) Cf. I, (2·4).

([24]) That s/c is equal to the point particle proper time when $\varkappa$ is normalized as in § 1 can be inferred from (4.5) in exactly the same way as this was done in § 1 after eq. (1.14).

have is one in which a non-zero $f_{\alpha\beta}$ would describe the Coulomb field due to a charge on the source. But the standard treatment ([25]) of the ad hoc Einstein-plus-Maxwell theory leading to the solution

$$f_i = 0 \quad (i = 1, 2, 3), \quad f_4 = e/r ; \qquad \gamma = 1 - 2m/r + 4\pi e^2/r^2 ,$$

where γ^{-1} and $-\gamma$ are the coefficients of dr^2 and $(dx^4)^2$ resp., fails here. The analytical reason is that whereas the gravitational and Maxwell energy tensors are coupled by a constant (Einstein gravitational constant) in the old theory, in this theory that coefficient is proportional to $(x^5)^2$ (vide eq. (A.21)). From this fact it can be easily shown that no exact solution with $g_{\alpha\beta}$ and non-zero $f_{\alpha\beta}$ both independent of x^5 exists. On the other hand the correct x^5-dependent solution is complicated by the fact that we now have partial differential equations (in r and x^5) to solve.

Appendix I

With only $k_{5m} \equiv \psi_m \neq 0$ we compute various relevant expressions. First, then $\mu_{5m} = \psi_m$, other $\mu_{\alpha\beta} = 0$;

$$\begin{aligned}
\tau_m &= -\varepsilon\partial_5\psi_m , \qquad \tau_5 = \partial_m\psi^m ; \\
H_{mn} &= -\tfrac{1}{2}(\partial_m\tau_n + \partial_n\tau_m) - \varepsilon\delta_{mn}\partial_5(\partial_p\psi^p) , \\
H_{5m} &= \tfrac{1}{2}\nabla^2\psi_m - \tfrac{1}{2}(\partial_m\tau_5 + \partial_5\tau_m) , \\
H_{55} &= -\partial_5\tau_5 + \partial_5(\partial_p\psi^p) ; \\
M_{mn} &= -(\varepsilon/2)(\partial_m\psi_n + \partial_n\psi_m - 2\delta_{mn}\tau_5) , \\
M_{5m} &= 0 , \\
M_{55} &= \tau_5 .
\end{aligned}$$

Hence the field equations (1.4) b) are satisfied identically and the set a run

$$\begin{aligned}
&(\partial_5 - 3\lambda^{-1})(\partial_m\psi_n + \partial_n\psi_m - 2\delta_{mn}\partial_p\psi^p) = 0 , \\
&(\Box^2 - \varepsilon\partial_5^2)\psi_m - (\partial_m(\partial_r\psi^r) - \varepsilon\partial_5^2\psi_m) = 0 , \\
&\partial_p\psi^p = 0 ,
\end{aligned}$$

whence the eq. (2.1) follow.

([25]) A. Eddington: *The Mathematical Theory of Relativity* (Cambridge, 1937), p. 185.

APPENDIX II

We get first

$$\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}_g - (x^5)^{-1}(\delta^5_\beta \delta^\alpha_\gamma + \delta^5_\gamma \delta^\alpha_\beta - g^{5\alpha} g_{\beta\gamma}) ,$$

where $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}_g$ is the Christoffel symbol of the $g_{\alpha\beta}$. The gives for the Einstein tensor $G_{\alpha\beta}$

$$G_{\alpha\beta} = \overline{G}_{\alpha\beta} + 3(x^5)^{-1} U_{\alpha\beta} + 6\varepsilon(x^5)^{-2} g_{\alpha\beta} ,$$

where $U_{\alpha\beta} = \left\{ \begin{matrix} 5 \\ \alpha\beta \end{matrix} \right\}_g - g_{\alpha\beta} \left\{ \begin{matrix} 5 \\ \gamma\delta \end{matrix} \right\} g^{\gamma\delta}$ and $\overline{G}_{\alpha\beta}$ is the Einstein tensor of $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}_g$. For the Maxwell energy tensor $T_{\alpha\beta}$ we have

$$T_{\alpha\beta} = \varepsilon(x^5)^2 \overline{T}_{\alpha\beta} ,$$

where $\overline{T}_{\alpha\beta}$ is the corresponding tensor with indices raised with $g^{\alpha\beta}$ rather than $\gamma^{\alpha\beta}$. Putting these into (4.1) a) we get

$$\overline{G}_{\alpha\beta} + (\varepsilon/2)(x^5)^2 \overline{T}_{\alpha\beta} + 3(x^5)^{-1} U_{\alpha\beta} = 0 . \tag{A2.1}$$

For the second and third set, we get at once

$$\partial_\alpha [(x^5)^{-1} |g|^{\frac{1}{2}} f^{\underline{\alpha}\beta}] = 0 ; \quad \partial_{\langle\alpha} f_{\beta\gamma\rangle} = 0 , \tag{A2.2}$$

where for the underlined upper index cf. (4.5). The eq. (4.1) are a linearization of (A2.1), (A2.2).

Now for a solution (4.6), $T_{\alpha\beta} = 0$ and (A2.2) are identically satisfied. Moreover all the $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}_g$ with any index 5 vanish, and the rest are functions only only of x^ρ. Hence $U_{\alpha\beta} = 0$, $\overline{G}_{5\alpha} = 0$, and (A2.1) reduces to

$$\overline{G}_{mn} = 0 .$$

RIASSUNTO (*)

Si dimostra che una particella classica di dimensioni finite è associabile con il nuovo concetto di particella implicito nell'uso della geometria conforme delle sfere dello spazio-tempo. L'introduzione della particella finita è covariante non solo secondo LORENTZ ma anche conformalmente. La relazione massa-energia per queste particelle finite libere

(*) *Traduzione a cura della Redazione.*

contiene un termine aggiuntivo che generalizza la relazione massa-energia data da EINSTEIN in relatività speciale per le particelle puntiformi. Nella rappresentazione (posizione) coniugata si incontra un termine aggiuntivo simile nella relazione tempo proprio-tempo data da EINSTEIN per le particelle puntiformi che estende la definizione di tempo proprio alle particelle finite. In § **1** si dànno equazioni di campo e di moto conformalmente invarianti di forma per il caso del campo debole. Lateoria di Maxwell nel vuoto ne è un caso speciale. Si dimostra che gli effetti finiti sono di origine elettromagnetica; ad esempio, la dimensione della particella finita è il « raggio classico » definito dal suo rapporto carica-massa. In § **2** si esaminano le soluzioni gravitazionali « estese ». Comprendono come caso speciale la teoria di Einstein della gravitazione debole. Si dimostra che queste forze sono indipendenti dalla carica. In § **3** si esamina la connessione dei termini ultramaxwelliani con lo smorzamento della radiazione nelle soluzioni elettromagnetiche « estese ». Si discutono varie nuove possibilità di eliminare le difficoltà che sorgono dall'« auto-forza » della particella puntiforme. In § **4** si rilevano alcuni aspetti delle equazioni esatte proposte. Si dimostra che la relatività generale è contenuta nella relatività conforme, nel senso che ogni soluzione esatta della prima dà una soluzione esatta « puramente gravitazionale » della seconda.

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The Scattering Operator and the Adiabatic Theorem (*).

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Summary. — It is shown that the outgoing and incoming eigenfunctions which are used in scattering theory can be obtained by specifying a relationship between the initial or final states and the state at time $t=0$. Thus appeal to the adiabatic theorem is unnecessary.

1. – Introduction.

In the older treatments of scattering problems the eigenfunctions of the total Hamiltonian corresponding to the continuous spectrum were specified in a time-independent way by requiring « outgoing wave » conditions in terms of the x-representation. This is the procedure used, for example, by Dirac [1] and Mott and Massey [2]. More recent writers, in trying to free themselves from imposing conditions in terms of the x-representation, have considered the initial value problem associated with the time-dependent Schrödinger equation. The solution is prescribed to be an eigenfunction of the unperturbed Hamiltonian in the infinite past and then the outgoing eigenfunctions are defined as being the solution of the time-dependent Schrödinger equation when the time t equals zero. To obtain sensible results most of these writers resort to the device of assuming that the perturbation is switched off adiabatically

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(1) P. A. M. Dirac: *The Principles of Quantum Mechanics*, Third Edition (London, 1947), Chapter VIII, § 49.

(2) N. F. Mott and H. S. W. Massey: *The Theory of Atomic Collisions*, Second Edition (London, 1949).

as the time t approaches $-\infty$. This is the procedure used, for example, by LIPPMANN and SCHWINGER ([3]) and GELL-MANN and GOLDBERGER ([4]).

However, FRIEDRICHS ([5]) has shown that if one works with an appropriate class of solutions of the time-dependent equation, namely those which are *wave packets* of the eigenfunctions corresponding to the continuous spectrum of the total Hamiltonian, the adiabatic switch-off of the perturbation is unnecessary. This fact has been called «the irrelevance of the adiabatic hypothesis» by SNYDER ([6]) who has considered the problem from a different point of view.

In the present paper we use Friedrich's approach to derive expressions for the «outgoing and incoming» eigenfunctions and the scattering operator arising from the time-dependent Schrödinger equation. The use of adiabatic switch-off procedures will be found unnecessary and the mathematical procedures will appear more natural then heretofore.

For the sake of completeness we shall include a discussion of the adiabatic approach in order to compare it with the procedure used herein.

2. – Definition of the Scattering Operator.

The introduction of the scattering operator is perhaps best motivated in the following fashion. Let us consider the Schrödinger equation which must be satisfied by the state vector $\Phi(t)$.

$$-\frac{1}{i}\frac{\partial}{\partial t}\Phi(t) = H\Phi(t)\,, \qquad (\hbar = 1) \tag{2.01}$$

where H is the Hamiltonian of the system. The equation for the state $\Phi(t)$ is of first order in time. Hence $\Phi(t)$ is uniquely determined, when it is prescribed for one value of time. One could, for example, take $\Phi(t)$ as being given at time $t = t_0$. Then $\Phi(t)$ is determined uniquely in terms of $\Phi(t_0)$.

In the scattering operator formalism one is interested in the case where the time t_0 is made to approach $-\infty$ and where t approaches $+\infty$. That is, one prescribes that state at a time infinitely distant in the past and seeks the state at a time infinitely distant in the future. The physical situation corresponding to such a calculation is one in which the length of time between the preparation of a state and the measurement of the state after an «exper-

([3]) B. LIPPMANN and J. SCHWINGER: *Phys. Rev.*, **79**, 469 (1950).
([4]) M. GELL-MANN and M. L. GOLDBERGER: *Phys. Rev.*, **91**, 398 (1953).
([5]) K. O. FRIEDRICHS: *Comm. on Pure and. App. Math.*, **1**, 361 (1948).
([6]) H. S. SNYDER: *Phys. Rev.*, **83**, 1154 (1951).

iment » is very long compared to the natural periods occuring in the system. In practice the physical situation is generally of this type. The advantage, as far as theory goes, of considering such situations is that the various expressions which one obtains for the state become considerably simpler.

In scattering experiments one thinks of the system as being separated into two parts corresponding to the following splitting up of the Hamiltonian:

$$H = H_0 + \varepsilon V . \tag{2.02}$$

Here H_0 is a time-dependent « unperturbed » Hamiltonian, ε is a smallness parameter, and εV is the « perturbation ».

In scattering problems this separation corresponds to the requirement that physically interesting solutions $\Phi(t)$ of (2.01) shall also be solutions, in some sense, of the unperturbed Schrödinger equation

$$-\frac{1}{i}\frac{\partial}{\partial t}\Phi(t) = H_0\Phi(t) \tag{2.03}$$

as $t \to \pm\infty$. This requirement means that as $t \to \pm\infty$, the influence of the perturbation εV is to disappear. Thus we shall prescribe solutions $\Phi(t)$ of the perturbed equation (2.01) to behave as specified solutions of the unperturbed equation (2.03) as $t \to -\infty$ and find the solutions $\Phi(t)$ as $t \to +\infty$ under the restriction that $\Phi(t)$ shall also behave as solutions of equation (2.03) $t \to +\infty$.

This way of looking at things appears as intuitively reasonable if one considers the problem of a particle which is scattered by a potential which vanishes outside some finite region in space. When $t \to -\infty$ one may think of the particle as being far from the scattering region so that the state vector obeys the unperturbed Schrödinger equation (2.03). When t assumes finite values the particle will be in the region where the potential exerts an influence so that the state vector satisfies the perturbed Schrödinger equation (2.01). As $t \to +\infty$, the particle passes into a region where the potential again has no effect and the state again is a solution of the unperturbed Schrödinger equation (2.03). The scattering operator is defined in such a manner as to give the dependence of the solution for $t = +\infty$ upon the solution at $t = -\infty$.

Influenced by such considerations we shall now define the scattering operator. Consider states $\Phi(t)$ which are solutions of (2.01) such that following limits exist:

$$\lim_{t \to \pm\infty} \exp[iH_0 t]\Phi(t) = \Phi_\pm . \tag{2.04}$$

Then the scattering operator S is defined by the relation

$$\Phi_+ = S\Phi_- . \tag{2.05}$$

If the limits (2.04) exist, we have obtained $\Phi(t)$ in terms of $\Phi(t_0)$ as $t \to +\infty$ and $t_0 \to -\infty$ as desired, for we can write

$$(2.06) \qquad \begin{cases} \Phi(t) \cong \exp[-iH_0 t]\Phi_+ , \\ \qquad \cong \exp[-iH_0 t] S \Phi_- , \\ \qquad \cong \exp[-iH_0 t] S \exp[iH_0 t_0] \Phi(t_0) . \end{cases}$$

In practice one prescribes Φ_- (which is equivalent to prescribing $\Phi(t_0)$ for $t_0 \to -\infty$) and by means of the scattering operator obtains Φ_+ (which is equivalent to obtaining $\Phi(t)$ for $t \to +\infty$). It will not be necessary to use (2.06) explicitly.

In the present paper we shall consider the case where the perturbation is constant in time. In addition, we restrict ourselves to Hamiltonians H and H_0 such that the spectrum of H_0 is purely continuous and is bounded from below, and the spectrum of H consists of a continuous part which coincides with the spectrum of H_0 and possible point eigenvalues which lie below the continuous spectrum. Though these restrictions on H and H_0 appear to be severe, many of the results can be generalized immediately to the case where H and H_0 suffer the less severe restrictions that their spectra coincide in a range where they are both continuous. It should be pointed out that the above restrictions on the spectra of H and H_0 are implicit in the usual treatment of scattering operators. However, we make a point of indicating such restrictions explicitly.

3. – The Spectral Representation of the Unperturbed Hamiltonian H_0.

We should like to set up a representation in terms of the unperturbed Hamiltonian H_0. This representation is convenient because in this representation the expression $\exp[iH_0 t]\Phi(t)$ of equation (2.04) takes a particularly simple form. Moreover, it is customary to prescribe Φ_- and to ask for Φ_+ in this representation.

Since, in general, H_0 will have a degenerate spectrum, it is necessary to assume that there exist other operators which commute with H_0 and which, together with H_0, form a complete set of commuting variables. We shall designate the quantum number associated with H_0 by E and the quantum numbers associated with the other operators collectively by α. The quantum number E lies in a continuum $E_a \leqslant E \leqslant E_b$. In many problems E_b may be taken as $+\infty$. E_a will always be finite in what follows.

To give a more physical meaning we may consider the scattering of a partial by a potential in which case α would consist of the polar angles θ, φ

which give the direction of the momentum and which could also include discrete variables such as spin.

Let us designate the (improper) eigenvectors or eigenstates of H_0 by $\omega_0(E, \alpha)$. Then the eigenvectors $\omega_0(E, \alpha)$ satisfy the equation

$$H_0\omega_0(E, \alpha) = E\omega_0(E, \alpha) \tag{3.01}$$

and also satisfy the orthogonality relation

$$(\omega_0(E, \alpha), \omega_0(E', \alpha')) = \delta(E - E')\,\delta(\alpha, \alpha'), \tag{3.02}$$

where the left-hand side of (3.02) denotes the usual Hermitian inner product of $\omega_0(E, \alpha)$ and $\omega_0(E', \alpha')$. The function $\delta(E - E')$ is the Dirac δ-function, and $\delta(\alpha, \alpha')$ is a generalized δ-function defined by

$$\begin{cases} \int_R r(\alpha)\,\delta(\alpha, \alpha')\,\mathrm{d}\alpha = r(\alpha') & \text{if the region of integration } R \text{ includes } \alpha' \\ = 0 & \text{otherwise.} \end{cases} \tag{3.03}$$

Integrations over α are to be interpreted as summations over the discrete variables included in α.

Let us expand any state vector Φ as follows

$$\Phi = \int_{E_a}^{E_b}\int f(E, \alpha)\omega_0(E, \alpha)\,\mathrm{d}\alpha\,\mathrm{d}E\,. \tag{3.04}$$

From (3.02) the coefficients $f(E, \alpha)$ of the expansion are given by

$$f(E, \alpha) = (\omega_0(E, \alpha), \Phi). \tag{3.05}$$

The function $f(E, \alpha)$ is the representative of Φ in the H_0 representation and we write

$$\Phi \underset{E}{\longleftrightarrow} f(E, \alpha) \tag{3.06}$$

to indicate the one-to-one correspondence between the state Φ and its H_0 representative.

We note that

$$H_0\Phi = \int_{E_a}^{E_b}\int Ef(E, \alpha)\omega_0(E, \alpha)\,\mathrm{d}\alpha\,\mathrm{d}E, \tag{3.07}$$

so that we may write

$$H_0\Phi \underset{E}{\longleftrightarrow} Ef(E, \alpha) . \tag{3.08}$$

If to every state Φ we can uniquely assign a function $f(E, \alpha)$ such that the function $Ef(E, \alpha)$ is assigned to the state $H_0\Phi$, we shall call such an assignment a spectral representation of the operator H_0.

We note that for any function of H_0, $L(H_0)$ say, we have

$$L(H_0)\Phi \underset{E}{\longleftrightarrow} L(E)f(E, \alpha) . \tag{3.09}$$

In fact, in a really rigorous treatment, (3.09) is essentially used to *define* $L(H_0)$. In particular we have

$$\exp[iH_0t]\Phi \underset{E}{\longleftrightarrow} \exp[iEt]f(E, \alpha) . \tag{3.10}$$

The inner product of two states $\Phi^{(1)}$, Φ can be written as

$$(\Phi^{(1)}, \Phi) = \int_{E_a}^{E_b}\int \bar{f}^{(1)}(E, \alpha)f(E, \alpha)\,\mathrm{d}\alpha\,\mathrm{d}E, \tag{3.11}$$

where $f^{(1)}$ and f are the representatives of $\Phi^{(1)}$ and Φ respectively. Equation (3.11) follows from the orthogonality relations (3.02). Throughout this report, unless otherwise noted, we shall restrict ourselves to proper states. That is, we shall ɔsn as representatives $f(E, \alpha)$ only functions which are quadratically integrable unless we specifically state otherwise. The use of such proper states simplifies much of our work. Since these states are those which are observed in nature, the motivation of much of what follows is made easier, and we are able to avoid some of the artificialities which one is forced to resort to if one works with improper states. An eigenstate of H_0 corresponding to the eigenvalues E', α' has the following H_0-representer

$$\omega_0(E', \alpha') \underset{E}{\longleftrightarrow} (\omega_0(E, \alpha), \omega_0(E', \alpha')) = \delta(E - E')\delta(\alpha, \alpha') . \tag{3.12}$$

We shall consider this function of E and α as being a limiting form of a representer $f(E, \alpha)$ which has a sharp peak at E', α'.

We shall designate any operator K in the H_0 by K^E. By definition, if $\Phi \underset{E}{\longleftrightarrow} f(E, \alpha)$, then

$$K\Phi \underset{E}{\longleftrightarrow} K^E f(E, \alpha) . \tag{3.13}$$

Note that from (3.08) we have

$$H_0^E = E . \tag{3.08a}$$

Usually it will be convenient to express K^E as an integral operator in the following way:

$$K^E f(E, \alpha) = \int_{E_a}^{E_b} \int K^E(E, \alpha | E', \alpha') f(E', \alpha') \, d\alpha' \, dE' . \tag{3.14}$$

The kernel $K^E(K, \alpha | E', \alpha')$ of the integral operator is just the usual « matrix » of K in terms of the H_0 representation, i.e.,

$$K^E(E, \alpha | E', \alpha') = (\omega_0(E, \alpha), K\omega_0(E', \alpha')) . \tag{3.15}$$

4. – The Spectral Representation of the Total Hamiltonian H.

Since the solution $\Phi(t)$ of the Schrödinger equation (2.01) is most simply expressed in terms of the spectral representation of the total Hamiltonian H, we shall introduce this representation. Let us designate the quantum number associated with H by F. As mentioned earlier, we assume that the spectrum of H consists of a continuous part which coincides with the spectrum of H_0 and point eigenvalues which lie below the continuous spectrum. Hence F lies in a continuum when $E_a \leqslant F \leqslant E_b$ and consists of points F_i when $F < E_a$. Furthermore, since we have assumed that the continuous spectrum has the same degeneracy as the spectrum of H_0, we shall have to introduce auxiliary quantum numbers β which are analogous to α and which have the same range. For the bound states or point eigenvalues the degeneracy may have any character whatsoever. We shall introduce auxiliary quantum numbers also denoted by β to describe the degeneracy of the spectrum of the bound states, though for these states β may have an entirely different character than when F lies in the continuum.

We can now introduce eigenvectors or eigenstates of H corresponding to the eigenvalue F and denote these eigenstates by $\omega(F, \beta)$ where F may either lie in the continuum $E_a \leqslant F \leqslant E_b$ or be one of the point eigenvalues F_i. We have

$$H\omega(F, \beta) = F\omega(F, \beta) . \tag{4.01}$$

The eigenfunctions also satisfy the orthogonality relations

$$(\omega(F, \beta), \omega(F', \beta')) = \delta(F - F')\delta(\beta, \beta') . \tag{4.02}$$

The function $\delta(F - F')$ is to be interpreted as a Dirac δ-function when both F and F' lie in the continuous spectrum, a Kronecker δ when both F,

F' lie in the point spectrum, and zero otherwise.

An arbitrary state Φ can be expanded in terms of the eigenfunctions $\omega(F, \beta)$ as follows

$$\Phi = \iint g(F, \beta)\omega(F, \beta)\, \mathrm{d}\beta\, \mathrm{d}F, \tag{4.03}$$

where the integration over the values of F and β corresponding to point eigenvalue are to be replaced by summation. We shall use this integration convention in order to minimize, in a formal fashion, the difference in character between the eigenstates of the continuous spectrum and those of the discrete spectrum.

From (4.02) and (4.03) we have

$$g(F, \beta) = (\omega(F, \beta), \Phi) \tag{4.04}$$

and we write

$$\Phi \underset{F}{\longleftrightarrow} g(F, \beta)\,. \tag{4.05}$$

We call $g(F, \beta)$ the representative of Φ in the H representation.

We note that

$$H\Phi \underset{F}{\longleftrightarrow} Fg(F, \beta) \tag{4.06}$$

and in general that a function of the operator H, $L(H)$ say, has the property

$$L(H)\Phi \underset{F}{\longleftrightarrow} L(F)f(F, \beta)\,. \tag{4.07}$$

In particular equation (4.07) gives us

$$\exp[-iHt]\Phi \underset{F}{\longleftrightarrow} \exp[-iFt]g(F, \beta)\,. \tag{4.08}$$

In terms of the spectrum representation of H, the inner product of two states $\Phi^{(1)}$, Φ whose representatives are $g^{(1)}(F, \beta)$ and $g(F, \beta)$ respectively is given by

$$(\Phi^{(1)}, \Phi) = \iint \bar{g}^{(1)}(F, \beta)g(F, \beta)\, \mathrm{d}\beta\, \mathrm{d}F, \tag{4.09}$$

where, in accordance with our previously mentioned convention, integrations over the quantum variables are to be understood as summations over those values which are point eigenvalues.

As in the case of the H_0 representation, we say that if to every state a unique assignment can be made to a function $g(F, \beta)$ such that $H\Phi$ is assigned

to $Fg(F, \beta)$, then we say that the set of functions $g(F, \beta)$ constitutes a spectral representation of the operator H.

We can also introduce operators in the H representation. The operator K in the H representation will be denoted by K^F which is defined by

$$K\Phi \underset{F}{\longleftrightarrow} K^F g(F, \beta) , \tag{4.10}$$

where

$$\Phi \underset{F}{\longleftrightarrow} g(F, \beta) .$$

5. – The Transformation from the H to the H_0 Representation.

It is desired to find the transformation which relates the H representer of an arbitrary state Φ to the H_0 representer of that state. The reason for wanting this transformation is that we can solve for the solution $\Phi(t)$ of the Schrödinger equation (2.01) in terms of the H representation in a simple way. We can then transform to the H_0 representation and thereby obtain explicitly the limits which occur in equation (2.04).

To be more specific let us take, as representatives of an arbitrary state

$$\Phi \underset{E}{\longleftrightarrow} f(E, \alpha)$$

$$\Phi \underset{F}{\longleftrightarrow} g(F, \beta)$$

and express the desired transformation as an integral transformation with the kernel $u(E, \alpha | F, \beta)$

$$f(E, \alpha) = \iint u(E, \alpha | F, \beta) g(F, \beta) \, \mathrm{d}\beta \, \mathrm{d}F . \tag{5.01}$$

Then if we write the representatives of the solution $\Phi(t)$ of the Schrödinger equation (2.01) as

$$\Phi(t) \underset{F}{\longleftrightarrow} g(F, \beta; t) \tag{5.02}$$

$$\Phi(t) \underset{E}{\longleftrightarrow} f(E, \alpha; t) , \tag{5.03}$$

we have from (5.01)

$$f(E, \alpha; t) = \iint u(E, \alpha | F, \beta) g(F, \beta; t) \, \mathrm{d}\beta \, \mathrm{d}F . \tag{5.04}$$

But the solution of (2.01) may be written as

$$\Phi(t) = \exp[-iHt]\Phi(0), \tag{5.05}$$

or in terms of the H representation (see (4.08))

$$g(F, \beta; t) = \exp[-iFt] g(F, \beta; 0), \tag{5.06}$$

so that from (5.04) we have

$$f(E,\alpha;t) = \iint u(E,\alpha|F,\beta)\exp[-iFt]g(F,\beta;0)\,\mathrm{d}\beta\,\mathrm{d}F\,. \tag{5.07}$$

Now in the H_0 representation $\exp[iH_0t]\Phi(t)$ is represented by

$$\exp[iH_0t]\,\Phi(t) \underset{E}{\longleftrightarrow} \exp[iEt]f(E,\alpha;t) = \tag{5.08}$$

$$= \iint u(E,\alpha|F,\beta)\exp[-i(F-E)t]g(F,\beta;0)\,\mathrm{d}\beta\,\mathrm{d}F\,.$$

Thus if we know the kernel $u(E,\alpha|F,\beta)$, we can evaluate explicitly the limits $\lim_{t\to\pm\infty}\exp[iH_0t]\Phi(t)$ which occur in (2.04) in terms of the H_0 representation. If these limits exist, we can find $\Phi_\pm$ and the scattering operator in the H_0 representation. We shall therefore examine the properties of the kernel $u(E,\alpha|F,\beta)$.

It should be noted that $u(E,\alpha|F,\beta)$ is a symbolic function rather than a proper function. For example, if the perturbation is identically zero, the functions $f(E,\alpha)$ and $g(F,\beta)$ of equation (5.01) must be identical and $u(E,\alpha|F,\beta)$ must be a δ-function. MØLLER (7) calls a function equivalent to $u(E,\alpha|F,\beta)$ « the wave matrix ». FRIEDRICHS (5) interprets (5.01) as being the result of an operation in Hilbert space rather than a transformation between two representations of Hilbert space.

Perhaps a more convenient way of looking at $u(E,\alpha|F,\beta)$ is to recognize it as an eigenfunction of H, corresponding to the eigenvalues F, β as given in the H_0 representation, i.e.,

$$\omega(F,\beta) \underset{E}{\longleftrightarrow} u(E,\alpha|F,\beta)\,, \tag{5.09}$$

or equivalently

$$u(E,\alpha|F,\beta) = (\omega_0(E,\alpha),\,\omega(F,\beta))\,. \tag{5.10}$$

Equation (5.10) and hence equivalently (5.09) follows from the equality

$$\int_{E_a}^{E_b}\!\!\int f(E,\alpha)\omega_0(E,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}E = \iint g(F,\beta)\omega(F,\beta)\,\mathrm{d}\beta\,\mathrm{d}F$$

where

$$\Phi \underset{E}{\longleftrightarrow} f(E,\alpha)$$

$$\Phi \underset{F}{\longleftrightarrow} g(F,\beta)\,,$$

(7) C. MØLLER, *General Properties of the Characteristic Matrix im the Theory of Elementary Particles* in *Det. Kgl. Danske Videnskabernes Selskab*, (København, 1945).

the state Φ being arbitrary. The use of the orthogonality relation (3.02) for $\omega_0(E, \alpha)$ yields (5.01) where $u(E, \alpha | F, \beta)$ is given by (5.10).

6. – Characterization of the Transformation.

We want to find an equation for the kernel $u(E, \alpha | F, \beta)$ of the transformation. Since $u(E, \alpha | F, \beta)$ is an eigenfunction of H in the H_0 representation corresponding to the eigenvalue F, we must have

$$H^E u(E, \alpha | F, \beta) = F u(E, \alpha | F, \beta) , \tag{6.01}$$

where H^E is the operator H as given in the H_0 representation. The operator H^E operates on the variables E, α of $u(E, \alpha | F, \beta)$. Now,

$$\begin{aligned} H^E &= H_0^E + \varepsilon V^E \\ &= E + \varepsilon V^E . \end{aligned} \tag{6.02}$$

Hence (6.01) becomes

$$(F - E) u(E, \alpha | F, \beta) = \varepsilon V^E u(E, \alpha | F, \beta) . \tag{6.03}$$

Equation (6.03) must be satisfied by all eigenfunctions of H. However, the eigenfunctions are not completely characterized by this equation. The eigenfunctions corresponding to eigenvalues F lying in the continuum $E_a \leqslant F \leqslant E_b$ must have further conditions imposed upon them, as we shall now see.

However, let us first consider equation (6.03) when F is one of the point eigenvalues F_i. The factor $(F_i - E)$ on the left-hand side of (6.03) is nowhere zero, since all values of E lie above F_i. Hence we can divide both sides of (6.03) by $F_i - E$ for all values of E. Hence we obtain for the eigen-functions belonging to point eigenvalues of H the following equation

$$u(E, \alpha | F_i, \beta) = \frac{\varepsilon}{F_i - E} V^E u(E, \alpha | F_i, \beta) . \tag{6.04}$$

It should be noted that $u(E, \alpha | F, \beta)$ is a proper, i.e., quadratically integrable, function of the variables E, α.

When F lies in the continuum spectrum, it has the same range of values as E. Hence the factor $F - E$ can be zero. Now generally when we wish to divide by a number which can have the value zero we have the possibility of getting symbolic functions. That is, if

$$x k(x) = r(x) , \tag{6.05}$$

when x can take the value zero, we can conclude that

$$k(x) = c\delta(x) + \frac{P}{x} r(x), \tag{6.06}$$

where c is an arbitrary constant and P/x means that for those integrals where $1/x$ appears as one of the factors of the integrand, the principal part of the integral is to be used (see, e.g., DIRAC [8]). That $k(x)$ as given (6.06) satisfies equation (6.05) can be verified readily.

From the above considerations we obtain from (6.03) the following expression for $u(E, \alpha | F, \beta)$ when F lies in the continuum spectrum

$$u(E, \alpha | F, \beta) = \lambda(E, \alpha | E, \beta)\, \delta(E - F) + \varepsilon \frac{P}{F - E} V^E u(E, \alpha | F, \beta), \tag{6.07}$$

or

$$u(E, \alpha | F, \beta) = \lambda(E, \alpha | E, \beta)\, \delta(E - F) + \varepsilon \frac{P}{F - E} \int_{E_a}^{E_b} \int V^E(E, \alpha | E', \alpha')\, u(E', \alpha' | F, \beta)\, d\alpha'\, dE', \tag{6.07a}$$

where $\lambda(E, \alpha | E, \beta)$ is an arbitrary function of its arguments. In order to determine λ we must now impose additional conditions on the eigenfunctions of H belonging to the continuous spectrum.

7. – Determination of the Eigenfunctions of the Continuous Spectrum.

We shall now determine the function $\lambda(E, \alpha | E, \beta)$ of equation (6.07). This function is usually obtained by following Dirac's method [9] in which the eigenfunctions $u(E, \alpha | E, \beta)$ are transformed to the $\boldsymbol{x}$ representation and boundary conditions are imposed on the eigenfunctions with respect to the variable $\boldsymbol{x}$. We shall follow a different procedure. In this procedure we specify a relationship between either Φ_- or Φ_+ of equation (2.04) and $\Phi(0)$.

Let us first note that for solutions $\Phi(t) = \exp[-iHt]\Phi(0)$ of equation (2.01), the limits $\lim_{t \to \pm\infty} \exp[iH_0 t]\Phi(t)$ occuring in (2.04) exist only if $\Phi(0)$ — or equivalently $\Phi(t)$ — is orthogonal to the bound eigenstates of H. To show this let us write equation (5.08) which gives $\exp[iH_0 t]\Phi(t)$ in the H_0 repres-

[8] P. A. M. DIRAC: loc. cit., Chapter III, § 15.
[9] P. A. M. DIRAC: loc. cit., Chapter VIII, § 50.

entation as

$$(7.01)\quad \exp[iH_0t]\Phi(t) \overset{E}{\longleftrightarrow} \exp[iEt]f(E,\alpha;t) =$$

$$= \int_{E_a}^{E_b}\int u(E,\alpha|F,\beta)\exp[-i(F-E)t]g(F,\beta;0)\,d\beta\,dF +$$

$$+ \sum_i \int u(E,\alpha|F_i,\beta)E\exp[-i(F_i-E)t]g(F_i,\beta;0)\,d\beta\,.$$

In (7.01) we have broken up the formal integration over F into a true integration over the continuous spectrum and a summation over the discrete spectrum. We shall later show that as $t\to\pm\infty$ in (7.01) the integral approaches a limit for many functions $g(F,\beta;0)$ defined for $E_a\leqslant F\leqslant E_b$. However, the sum over the point eigenvalues, as can be seen, merely oscillates and does not approach a limit as $t\to\pm\infty$.

Hence, in order to have limits $\Phi_\pm$, we must take $g(F,\beta;0)=0$ whenever F belongs to the discrete spectrum. Since $g(F_i,\beta;0)=(\omega(F_i,\beta),\Phi(0))$ we have our results that $\Phi(0)$ must be orthogonal to the eigenfunctions of the discrete spectrum.

Let us therefore now restrict ourselves to state $\Phi(0)$ such that its H representer $g(F,\beta;0)$ is identically zero for values of F corresponding to the discrete spectrum. We shall now characterize the states Φ_- or Φ_+ in terms of $\Phi(0)$.

It is clear that the H_0 representer of $\Phi_\pm$ will be linearly related to the H representer of $\Phi(0)$. Furthermore it is clear that

$$(7.02)\qquad (\Phi_+,\Phi_+)=(\Phi_-,\Phi_-)=(\Phi(0),\Phi(0))$$

in order that probability may be preserved. Let us express $\Phi(0)$ in the H representation and $\Phi_\pm$ in the H_0 representation in the following way

$$\Phi(0)\overset{F}{\longleftrightarrow} g(F,\beta;0) \qquad (E_a\leqslant F\leqslant E_b)$$

$$\Phi_\pm \overset{E}{\longleftrightarrow} f_\pm(E,\alpha)\,.$$

Then one may write the relation between $f_\pm$ and g as an integral transformation

$$(7.03)\qquad f_\pm(E,\alpha)=\int_{E_a}^{E_b}\int K_\pm(E,\alpha|F,\beta)\,g(F,\beta;0)\,d\beta\,dF\,,$$

where $K_{\pm}$ satisfy the orthogonality relations

$$(7.04)\quad \begin{cases} \int_{E_a}^{E_b}\int \bar{K}_{\pm}(E,\alpha\,|\,F,\beta)K_{\pm}(E,\alpha\,|\,F',\beta')\,\mathrm{d}\alpha\,\mathrm{d}E = \delta(F-F')\,\delta(\beta,\beta')\,, \\ \int_{E_a}^{E_b}\int \bar{K}_{\pm}(E,\alpha\,|\,F,\beta)K_{\pm}(E',\alpha'\,|\,F,\beta)\,\mathrm{d}\beta\,\mathrm{d}F = \delta(E-E')\,\delta(\alpha,\alpha')\,, \end{cases}$$

where in the above F and F' are restricted to the continuous spectrum of H. Each different way of choosing K_+ or K_- will lead to a different choice of eigenfunctions belonging to the continuous spectrum. We can prescribe *either* K_+ *or* K_-; we cannot choose them both.

Let us first consider the simplest choice of K_-, namely

$$(7.05)\qquad K_-(E,\alpha\,|\,F,\beta) = \delta(E-F)\delta(\alpha,\beta)$$

or from (7.03)

$$(7.06)\qquad f_-(E,\alpha) = g(E,\alpha;\,0)\,.$$

We may re-express condition (7.05) in the following way: *We shall prescribe the solution* $\Phi(t)$ *of the perturbed Schrödinger equation* (2.01) *to be in an asymptotic sense also a solution of the unperturbed Schrödinger equation* (2.03).

$$(7.07)\qquad \Phi(t) \underset{E}{\longleftrightarrow} \exp[-iEt]f_-(E,\alpha)\,.$$

We shall then have as a solution $\Phi(t)$ *of the perturbed equation* (2.01) *at finite times*

$$(7.08)\qquad \Phi(t) \underset{F}{\longleftrightarrow} g(F,\beta;\,t) = \exp[-iFt]f_-(F,\beta)\,.$$

It will not be possible to obtain **bound states** *of* H *from such initial conditions.*

In the problem where a particle is scattered by a potential, the function $f_-(E,\alpha)$ would be the assigned probability amplitude that the particle initially had the kinetic energy E and direction designated by α before collision. Even in the case of an attractive potential which is capable of having a bound state it is not possible for the particle to be captured.

We shall now show how (7.06) characterizes our eigenfunctions.

It is convenient to introduce the matrix $T(E,\alpha\,|\,F,\beta)$ defined by

$$(7.09)\qquad T(E,\alpha\,|\,F,\beta) = \int_{E_a}^{E_b}\int V^E(E,\alpha\,|\,E',\alpha')u(E',\alpha'\,|\,F,\beta)\,\mathrm{d}\alpha'\mathrm{d}E'\,.$$

The integral equation (6.07*a*) is then written

$$(7.10)\qquad u(E,\alpha|F,\beta)=\lambda(E,\alpha|E,\beta)\,\delta(E-F)+\varepsilon\frac{P}{F-E}\,T(E,\alpha|F,\beta)\,.$$

Equation (7.01) is then

$$(7.11)\qquad \exp[iEt]f(E,\alpha;t)=\int_{E_a}^{E_b}\!\!\int u(E,\alpha|F,\beta)g(F,\beta;0)\exp[-i(F-E)t]\,\mathrm{d}\beta\,\mathrm{d}F=$$

$$=\int\lambda(E,\alpha|E,\beta)g(E,\beta;0)\,\mathrm{d}\beta+$$

$$+\varepsilon\int_{E_a}^{E_b}\!\!\int\frac{T(E,\alpha|F,\beta)}{F-E}\,g(F,\beta;0)\,\exp[-i(F-E)t]\,\mathrm{d}\beta\,\mathrm{d}F\,.$$

In equation (7.11) and in the integrals which follow we drop the symbol P but understand that the principal values of the integrals containing the factor $1/(F-E)$ in the integrands are to be taken.

We shall choose λ by using our above-mentioned requirement that

$$(7.12)\qquad \lim_{t\to-\infty}\exp[iEt]f(E,\alpha;t)\equiv f_{-}(E,\alpha)=g(E,\alpha;0)\,.$$

Therefore, let us consider the second term of equation (7.11)

$$\lim_{t\to-\infty}\int_{E_a}^{E_b}\!\!\int\frac{T(E,\alpha|F,\beta)}{F-E}\,g(F,\beta;0)\exp[-i(F-E)t]\,\mathrm{d}\beta\,\mathrm{d}F\,.$$

To evaluate this limit we shall use the Riemann-Lebesgue theorem for Fourier Transforms which says that

$$(7.13)\qquad \lim_{\alpha\to\infty}\int_{A}^{B}f(x)\exp[i\alpha x]\,\mathrm{d}x=0\,,$$

if $f(x)$ is bounded and integrable in (A,B), or the integral $\int_{A}^{B}f(x)x$ is absolutely

convergent ([10]). We write

$$(7.14)\qquad \lim_{t\to-\infty}\int_{E_a}^{E_b}\int \frac{T(E,\alpha|F,\beta)}{F-E}\,g(F,\beta;0)\exp[-i(F-E)t]\,\mathrm{d}\beta\,\mathrm{d}F =$$

$$= \lim_{t\to-\infty}\int_{E_a}^{E_b}\int \frac{T(E,\alpha|F,\beta)g(F,\beta;0)-T(E,\alpha|E,\beta)g(E,\beta;0)}{F-E}\cdot$$

$$\cdot\exp[-i(F-E)t]\,\mathrm{d}F\,\mathrm{d}\beta + \lim_{t\to-\infty}\int T(E,\alpha|E,\beta)g(E,\beta;0)\,\mathrm{d}\beta\cdot$$

$$\cdot\int_{E_a}^{E_b}\frac{\exp[-i(F-E)t]}{F-E}\,\mathrm{d}F\,.$$

The expression

$$\frac{T(E,\alpha|F,\beta)g(F,\beta;0)-T(E,\alpha|E,\beta)g(E,\beta;0)}{F-E},$$

as a function of F is absolutely integrable for a large class of functions $g(F,\beta;0)$, if $T(E,\alpha|F,\beta)\,g(F,\beta;0)$ considered as a function of F has a derivative at $F=E$. Hence the first term on the right hand side of equation (7.14) vanishes for appropriate functions $g(F,\beta;0)$.

We shall now consider the second term of (7.14). We find by a suitable change of variable

$$(7.15)\qquad \lim_{t\to-\infty}\int_{E_a}^{E_b}\frac{\exp[-i(F-E)t]}{F-E}\,\mathrm{d}F = \lim_{t\to-\infty}\int_{(E_a-E)t}^{(E_b-E)t}\frac{\exp[-i\zeta]}{\zeta}\,\mathrm{d}\zeta\,.$$

Let us first take the case where $E_a<E<E_b$, that is where E lies *within* the limits of the spectrum of H_0. Equation (7.15) becomes

$$(7.16)\qquad \lim_{t\to-\infty}\int_{E_a}^{E_b}\frac{\exp[-i(F-E)t]}{F-E}\,\mathrm{d}F = \int_{+\infty}^{-\infty}\frac{\exp[-i\zeta]}{\zeta}\,\mathrm{d}\zeta = 2i\int_0^{\infty}\frac{\sin\zeta}{\zeta}\,\mathrm{d}\zeta = i\pi\,.$$

Hence we have from (7.11), (7.14), (7.16) and also our initial condition (7.12)

$$(7.17)\qquad \lim_{t\to-\infty}\exp[iEt]f(E,\alpha;t) = \int\lambda(E,\alpha|E,\beta)g(E,\beta;0)\,\mathrm{d}\beta +$$

$$+\,\varepsilon i\pi\int T(E,\alpha|E,\beta)g(E,\beta;0)\,\mathrm{d}\beta = g(E,\alpha;0)\,.$$

([10]) H. S. Carslaw: *Theory of Fourier Series and Integrals*, Third Edition (New York, 1945).

Thus from the definition of $\delta(\alpha, \alpha')$ (eq. (3.03))

$$\lambda(E, \alpha | E, \beta) + \varepsilon i\pi T(E, \alpha | E, \beta) = \delta(\alpha, \beta) , \tag{7.17a}$$

or

$$\lambda(E, \alpha | E, \beta) = \delta(\alpha, \beta) - i\varepsilon\pi T(E, \alpha | E, \beta) . \tag{7.18}$$

Our integral equation (6.07a) for $u(E, \alpha | F, \beta)$ is thus

$$u_-(E, \alpha | F, \beta) = \delta(E - F)\,\delta(\alpha, \beta) + \varepsilon\left[-i\pi\delta(F - E) + \frac{P}{F - E}\right] T_-(E, \alpha | F, \beta) , \tag{7.19}$$

or

$$u_-(E, \alpha | F, \beta) = \delta(E - F)\,\delta(\alpha, \beta) + \varepsilon\left[-i\pi\delta(F - E) + \frac{P}{F - E}\right] \cdot \int_{E_a}^{E_b}\int V^E(E, \alpha | E', \alpha') u_-(E', \alpha' | F, \beta)\, \mathrm{d}\alpha'\, \mathrm{d}'E . \tag{7.19a}$$

In (7.19) and (7.19a) we have introduced the negative subscript to indicate that the eigenfunctions of the continuous spectrum have been characterized by initial conditions given at $t = -\infty$.

Having shown that the limit $\lim_{t \to -\infty} \exp[iEt] f(E, \alpha; t) = f_-(E, \alpha)$ exists when E lies within the end points of the spectrum of H_0, let us now consider the case where E equals one of the end points, $E = E_a$, say. In the development carried out above equation (7.15) is replaced by

$$\lim_{t \to -\infty} \int_{E_a}^{E_b} \frac{\exp[-i(F, E)t]}{F - E_a}\, \mathrm{d}F = \int_0^{-\infty} \frac{\exp[-i\zeta]}{\zeta}\, \mathrm{d}\zeta . \tag{7.15a}$$

The integral diverges. Therefore we have to put some requirement on $T(E, \alpha | F, \beta)$ or $g(F, \beta; 0)$ in order to obtain $\lim_{t \to -\infty} \exp[iE_a t] f(E_a, \alpha; t)$. Now the second term on the right hand side of equation (7.14) will cause no trouble for $E = E_a$, if either $T(E_a, \alpha | E_a, \beta) = 0$ or $g(E_a, \beta; 0) = 0$.

Using the definition of $T(E, \alpha | F, \beta)$ given by (7.09) the first condition implies a condition on $V^E(E_a, \alpha | E', \alpha')$, namely

$$V^E(E_a, \alpha | E', \alpha') = 0 . \tag{7.20}$$

The second condition limits the class of states Φ_- or $\Phi(0)$ which can be used.

Similarly when $E = E_b$ we have two possibilities, namely we can either take

$$V^E(E_b, \alpha | E', \alpha') = 0\,, \tag{7.21}$$

or take $g(E_b, \beta; 0) = 0$.

Then with either the above conditions on V or on $g(F, \beta; 0) = f_-(F, \beta)$ one obtains the limit of $\exp[-iEt]f(E, \alpha; t)$ as t goes to $-\infty$ for all values of E.

We shall then consider $u_-(E, \alpha | F, \beta)$ as given by (7.19) as being an appropriate set of eigenfunctions for the continuous spectrum, provided they are applied either to suitable states $g(F, \beta; 0)$ or the perturbation satisfies conditions (7.20) and (7.21).

The conditions on $g(F, \beta; 0)$ or on $V^E(E, \alpha | E', \alpha')$ will apply to all other processes in which time limits are considered and we shall not repeat these conditions.

Now we shall look at the eigenfunctions which we obtain when conditions at $t = +\infty$ are prescribed. In analogy to the derivation of (7.19) and (7.19a), we take $f_+(E, \alpha)$ to be $g(E, \alpha; 0)$ and find the following equations for the eigenfunctions, which we denote by $u_+(E, \alpha | F, \beta)$

$$u_+(E, \alpha | F, \beta) = \delta(E-F)\,\delta(\alpha, \beta) + \varepsilon\left[i\pi\delta(F-E) + \underset{F-E}{P}\right] T_+(E, \alpha | F, \beta)\,, \tag{7.22}$$

or

$$u_+(E, \alpha | F, \beta) = \delta(E-F)\,\delta(\alpha, \beta) + \varepsilon\left[i\pi\delta(F-E) + \frac{P}{F-E}\right] \cdot \int_{E_a}^{E_b}\int V^E(E, \alpha | E', \alpha')\, u_+(E', \alpha' | F, \beta)\, \mathrm{d}\alpha'\, \mathrm{d}E'\,. \tag{7.22a}$$

For convenience let us introduce a notation for the functions $[\pm i\pi\delta(F-E) + P/(F-E)]$ which occur in (7.19) and (7.22). Accordingly we set

$$\gamma_\pm(x) = \pm i\pi\delta(x) + \frac{P}{x}\,. \tag{7.23}$$

As is well known the functions $\gamma_\pm$ can be considered as being the limits of proper functions

$$\gamma_\pm(x) = \lim_{\zeta \to 0} \frac{1}{x \pm i\zeta} = \left[\pm i\pi\delta(x) + \frac{P}{x}\right], \tag{7.24}$$

ζ being positive.

To summarize: The eigenfunctions of H corresponding to the discrete spectrum satisfy the integral equation

$$(7.25)\quad u(E,\alpha|F_i,\beta)=\frac{\varepsilon}{F_i-E}\int_{E_a}^{E_b}\int V^E(E,\alpha|E',\alpha')u(E',\alpha'|F_i,\beta)\,\mathrm{d}\alpha'\,\mathrm{d}E'\,,$$

while for the eigenfunctions corresponding to the continuous spectrum we can choose among others either of the two sets of eigenfunctions $u_-(E,\alpha|F,\beta)$ or $u_+(E,\alpha|F,\beta)$. The eigenfunctions $u_\pm$ satisfy the following integral equation

$$(7.26)\quad u_\pm(E,\alpha|F,\beta)=\delta(E-F)\delta(\alpha,\beta)+\varepsilon\gamma_\pm(F-E)T_\pm(E,\alpha|F,\beta)\,,$$

where

$$(7.27)\quad T_\pm(E,\alpha|F,\beta)=V^E u_\pm(E,\alpha|F,\beta)=$$
$$=\int_{E_a}^{E_b}\int V^E(E,\alpha|E',\alpha')u_\pm(E',\alpha'|F,\beta)\,\mathrm{d}E'\,\mathrm{d}\alpha'\,.$$

The eigenfunctions $u_-(E,\alpha|F,\beta)$ are used to construct solutions of the Schrödinger equation when initial conditions at $t=-\infty$ are given. When Φ_- is prescribed in the H_0 representation in the following way

$$\Phi_- \overset{E}{\longleftrightarrow} f_-(E,\alpha)\,,$$

then the solution $\Phi(t)$ in the H representation is given by

$$(7.28)\quad \Phi(t)\overset{F}{\longleftrightarrow} g(F,\beta;t)=\exp[-iFt]f_-(F,\beta)\,,$$

or in the H_0 representation by

$$(7.29)\quad \Phi(t)\overset{E}{\longleftrightarrow} f(E,\alpha;t)=\int_{E_a}^{E_b}\int u_-(E,\alpha|F,\beta)\exp[-iFt]f_-(F,\beta)\,\mathrm{d}\beta\,\mathrm{d}F\,.$$

The eigenfunctions $u_+(E,\alpha|F,\beta)$ are used to construct solutions of the Schrödinger equation when final conditions at $t=+\infty$ are prescribed. When Φ_+ is prescribed in the following way

$$\Phi_+ \overset{E}{\longleftrightarrow} f_+(E,\alpha)\,,$$

the solution $\Phi(t)$ is given by

$$(7.30)\quad \Phi(t)\overset{F}{\longleftrightarrow} g(F,\beta;t)=\exp[-iFt]f_+(F,\beta)\,,$$

or

$$\Phi(t) \underset{E}{\longleftrightarrow} f(E, \alpha; t) = \int_{E_a}^{E_b} \int u_+(E, \alpha | F, \beta) \exp[-iFt] f_+(F, \beta) \, d\beta \, dF . \tag{7.31}$$

It might be mentioned that instead of using the functions $\gamma_\pm$ MØLLER and others prefer to use the functions $\delta_\pm$ defined in the following way

$$\delta_\pm(x) = \mp \frac{i}{2\pi} \gamma_\pm(x) . \tag{7.32}$$

However, we find it more convenient to use the functions $\gamma_\pm$, which in effect are also used in reference (3) and (5). Incidently, the functions $u_\mp$ are the eigenstates $\Psi^\pm$ reference (3).

Let us now consider the case where K_+ or K_- is a more general operator than that of (7.05). Then from equations (7.03) and (7.12) we have as the initial conditions which determine the eigenfunctions u_-

$$\lim_{t \to -\infty} \exp[iEt] f(E, \alpha; t) = f_-(E, \alpha) = \int_{E_a}^{E_b} \int K_-(E, \alpha | F, \beta) g(F, \beta; 0) \, d\beta \, dF . \tag{7.33}$$

Equation (7.17a), the equation which determines $\lambda(E, \alpha | E, \beta)$ is replaced by

$$[\lambda(E, \alpha | E, \beta) + \varepsilon i\pi T(E, \alpha | E, \beta)] \, \delta(E - F) = K_-(E, \alpha | F, \beta) . \tag{7.34}$$

Equation (7.34) implies a restriction on the function K_- which can be used or, equivalently, a restriction on the relation which may be specified between Φ_- and $\Phi(0)$. From (7.34) and the unitarity relation (7.04) we see that $K_-(E, \alpha | F, \beta)$ must have the form

$$K_-(E, \alpha | F, \beta) = \delta(E - F) \exp[i\eta(E)] \mu(\alpha, \beta) , \tag{7.35}$$

where we may choose the functions $\eta(E)$ and $\mu(\alpha, \beta)$ in any way we wish, provided $\eta(E)$ is a real function of its argument and $\mu(\alpha, \beta)$ satisfies the orthogonality relations

$$\begin{cases} \int \bar{\mu}(\alpha, \beta) \mu(\alpha', \beta) \, d\beta = \delta(\alpha, \alpha') , \\ \int \bar{\mu}(\alpha, \beta) \mu(\alpha, \beta') \, d\alpha = \delta(\beta, \beta') . \end{cases} \tag{7.36}$$

With such a choice for K_- we find from (7.34)

$$\lambda(E, \alpha | E, \beta) = \exp[i\eta(E)]\mu(\alpha, \beta) - \varepsilon i\pi T(E, \alpha | E, \beta) . \tag{7.37}$$

Denoting the eigenfunctions u for which λ is chosen in this way by $u_{K_-}(E, \alpha | F, \beta)$ we obtain from (6.07a) the integral equation for these eigenfunctions

$$u_{K_-}(E, \alpha | F, \beta) = \exp[i\eta(E)]\mu(\alpha, \beta)\delta(E - F) + \gamma_-(F - E)\cdot \cdot \int V^E(E, \alpha | E', \alpha') u_{K_-}(E', \alpha' | F, \beta) \, \mathrm{d}E' \, \mathrm{d}\alpha' . \tag{7.38}$$

We thus obtain a set of eigenfunctions of H which is not usually considered explicitly in the literature. If we denote the eigenfunctions whose H_0 representatives are $u_-(E, \alpha | F, \beta)$ and $u_{K_-}(E, \alpha | F, \beta)$ by $\omega_-(F, \beta)$ and $\omega_{K_-}(F, \beta)$ respectively, it can be shown that ω_{K_-} can be obtained by the following relation from $\omega_-(F, \beta)$

$$\omega_{K_-}(F, \beta) = \exp[i\eta(F)] \int \mu(\beta', \beta)\omega_-(F, \beta') \, \mathrm{d}\beta' . \tag{7.39}$$

Hence all eigenfunctions of the continuous spectrum of H, obtained by specifying a relationship betweer $\Phi(0)$ and Φ_- can be expressed in terms of ω_-. There are many interesting consequences of (7.39) but we shall not pursue them further here.

8. – The Scattering Operator.

We are now in a position to express the scattering operator and its inverse in terms of the eigenfunctions u_- and u_+ respectively. Let us then use the eigenfunctions u_- and construct the solution of the Schrödinger equation $\Phi(t)$ in the H_0 representation, having prescribed the state Φ_-

$$\Phi(t) \underset{E}{\longleftrightarrow} f(E, \alpha; t) = \int_{E_a}^{E_b} \int u_-(E, \alpha | F, \beta) \exp[-iFt] f_-(F, \beta) \, \mathrm{d}\beta \, \mathrm{d}F , \tag{8.01}$$

where

$$\Phi_- \underset{E}{\longleftrightarrow} f_-(E, \alpha) .$$

Then we can evaluate the limit

$$\lim_{t \to +\infty} \exp[iEt] f(E, \alpha; t) = f_+(E, \alpha) =$$

$$= \lim_{t \to +\infty} \int_{E_a}^{E_b} \int u_-(E, \alpha | F, \beta) \exp[-i(F - E)t] f_-(F, \beta) \, \mathrm{d}\beta \, \mathrm{d}F .$$

by replacing $u_-(E, \alpha | F, \beta)$ by the right-hand side of (7.27) and treating the integral which contains the factor $P/(F - E)$ in a manner similar to that of the previous section. We find that

$$f_+(E, \alpha) = \int_{E_a}^{E_b}\int S^E(E, \alpha | E', \alpha') f_-(E', \alpha')\, \mathrm{d}\alpha'\, \mathrm{d}E\,, \tag{8.02}$$

where

$$S^E(E, \alpha | E', \alpha') = \delta(E - E')\delta(\alpha, \alpha') - 2\pi\varepsilon i\delta(E - E')T_-(E, \alpha | E', \alpha')\,. \tag{8.3}$$

Since $f_-(E, \alpha)$ and $f_+(E, \alpha)$ are the H_0 representatives of Φ_- and Φ_+ respectively, it is clear from the definition of the scattering operator (2.05) that $S^E(E, \alpha | E', \alpha')$ is the matrix of the scattering operator in the H_0 representation.

It might also be noted that $S^E(E, \alpha | F, \beta)$ is just the kernel $K_+(E, \alpha | F, \beta)$ of the transformation (7.03) when $K_-(E, \alpha | F, \beta)$ is given by (7.05). We should also note that it is not necessary that K_- be given by (7.05) in order to get an expression for the scattering operator. If we had chosen K_- to be the kernel of another unitary transformation, we should have obtained another representation of the scattering operator.

In a similar way, we construct the solution $\Phi(t)$ with the eigenfunctions u_+, and write

$$\Phi(t) \overset{E}{\longleftrightarrow} f(E, \alpha;\, t) = \int_{E_a}^{E_b}\int u_+(E,\, \alpha | F, \beta)\, \exp\left[-iFt\right] f_+(F, \beta)\, \mathrm{d}\beta, \mathrm{d}F\,, \tag{8.04}$$

where $f_+(E, \alpha)$ is a given function such that

$$\Phi_+ \overset{E}{\longleftrightarrow} f_+(E, \alpha)\,.$$

We can show

$$\lim_{t \to -\infty} \exp\left[iEt\right] f(E, \alpha;\, t) = f_-(E, \alpha) = \int_{E_a}^{E_b}\int \hat{S}^E(E, \alpha | E', \alpha')\, f_+(E', \alpha')\, \mathrm{d}\alpha'\, \mathrm{d}E', \tag{8.05}$$

where

$$\hat{S}^E(E, \alpha | E', \alpha') = \delta(E - E')\delta(\alpha, \alpha') + 2i\pi\varepsilon\delta(E - E')T_+(E, \alpha | E, \alpha')\,. \tag{8.06}$$

It is clear that $\hat{S}^E(E, \alpha | E', \alpha')$ is the matrix of the inverse scattering operator S^{-1} in the H_0 representation.

We can give $S^E(E, \alpha | E', \alpha')$ a more familiar interpretation in the following

way. In taking the limits as $t \to \pm\infty$ we have assumed that Φ_- or, equivalently, its H_0 representative $f_-(E, \alpha)$ in normalizable. However, having obtained expressions for the limits in time for such proper states, we can then take Φ_- to be an eigenstate of H_0, this being a state which is not normalizable and hence not really a physical state. We wan consider such an eigenstate as having been constructed from a proper state by a suitable limiting process. Let us then prescribe Φ_- to be an eigenstate of H_0 corresponding to the eigenvalues E', α'. Then $f_-(E, \alpha) = \delta(E - E')\delta(\alpha, \alpha')$ and $f_+(E, \alpha) = S^E(E, \alpha | E', \alpha')$. Hence the matrix of the scattering operator can be interpreted as the H_0 representer of the state after scattering has taken place. The first term of $S^E(E, \alpha | E', \alpha')$, namely $\delta(E - E')\,\delta(\alpha, \alpha')$, which is also equal to $f_-(E, \alpha)$, is called the « incident » or « inconing » wave. The second term is called the « scattered wave ».

9. – Adiabatic Switch-on of the Perturbation and the Adiabatic Theorem.

It has been shown that for a large class of solutions

$$\Phi(t) = \exp[-iHt]\Phi(0) \tag{9.01}$$

of the Schrödinger equation (2.01), one can evaluate the limits

$$\lim_{t \to \pm\infty} \exp[iH_0 t]\Phi(t) = \Phi_\pm . \tag{9.02}$$

Such solutions satisfy the unperturbed Schrödinger equation asymptotically as $t \to \pm\infty$ as pointed out in Section 2. The solutions thus behave as though the perturbation had been « switched off » as $t \to \pm\infty$. In particular it was shown by means of the eigenfunctions u_- that if the initial condition on the solution of the Schrödinger equation (2.01) were that the H_0 representative $f(E, \alpha; t)$ should behave like $\exp[-iEt]f_-(E, \alpha)$ at $t \to -\infty$, then at finite times the H representative $g(F, \beta; t)$ was given by

$$g(F, \beta; t) = \exp[-iFt]f_-(F, \beta)$$

this condition being, in fact the condition used to specify u_- completely.

We shall now consider the problem in which the perturbation is « really » switched off at a very slow rate, i.e., adiabatically, as $t \to -\infty$. We shall show that if we prescribed as the initial condition that the solution of the Schrödinger equation shall behave as above as $t \to -\infty$, then at finite times t the H_0 and H representative, $f(E, \alpha; t)$ and $g(F, \beta; t)$, respectively, are the

same as those obtained using the eigenfunctions u_- and assuming the perturbation constant in time.

SNYDER ([6]) appears to have been the first to have discussed the generally assumed equivalence of the adiabatic switch-off procedure and the procedure of using the spectral representation of H. FRIEDRICHS ([11]) has also discussed the equivalence of the adiabatic switch-off procedure to the procedure of using the spectral representation but in a more rigorous fashion.

In their discussion of the use of adiabatic switch off, both SNYDER and FRIEDRICHS have restricted themselves to the case where the total Hamiltonian at all times has a continuous spectrum only, which coincides with the spectrum of H_0. In our treatment we shall follow more closely the spirit of the work of LIPPMANN and SCHWINGER and FUBINI ([12]) in which no explicit assumption is made regarding the total Hamiltonian. However, in our discussion we shall indicate conditions needed to obtain meaningful results.

Accordingly we introduce the Hamiltonian $H_\zeta(t)$ defined by

$$H_\zeta(t) = H_0 + \varepsilon \exp[\zeta t] V \qquad (\zeta > 0). \tag{9.04}$$

We note that

$$\lim_{t \to -\infty} H_\zeta(t) = H_0 \tag{9.05}$$

and

$$\lim_{\zeta \to 0} H_\zeta(t) = H\,. \tag{9.06}$$

Let us denote by $f_\zeta(E, \alpha; t)$ the H_0 representative of $\Phi_\zeta(t)$ which is a solution of the Schrödinger equation

$$H_\zeta(t)\,\Phi_\zeta(t) = i\frac{\partial}{\partial t}\Phi_\zeta(t)\,. \tag{9.07}$$

The representative $f_\zeta(E, \alpha; t)$ thus satisfies the equation

$$i\frac{\partial f_\zeta(E, \alpha; t)}{\partial t} = E f_\zeta(E, \alpha; t) + \varepsilon \exp[\zeta t] \int_{E_a}^{E_b} \int V^E(E, \alpha \,|\, E', \alpha') f_\zeta(E', \alpha'; t)\, \mathrm{d}\alpha'\, \mathrm{d}E'\,. \tag{9.08}$$

We take as initial conditions

$$\lim_{t \to -\infty} \exp[iEt] f_\zeta(E, \alpha; t) = f_-(E, \alpha) \tag{9.09}$$

([11]) K. O. FRIEDRICHS: *Report on the Adiabatic Theorem*, New York University Report (in preparation).

([12]) S. FUBINI: *Nuovo Cimento*, **9**, 846 (1952).

where $f_-(E, \alpha)$ is prescribed. Let us write $f_\zeta(E, \alpha; t)$ in terms of $\exp[-iEt]f_-(E,\alpha)$, which is the solution of (9.08) if no perturbation were present, in terms of an integral operator as follows:

$$f_\zeta(E, \alpha; t) = \int_{E_a}^{E_b} \int w_{\zeta,t}(E, \alpha | E', \alpha') \exp[-iE't] f_-(E', \alpha') \mathrm{d}\alpha' \, \mathrm{d}E'. \tag{9.10}$$

It is our objective to show that in many cases $w_{\zeta,t}$ becomes the eigenfunction u_- in the adiabatic limit. That is,

$$\lim_{\zeta \to 0} w_{\zeta,t}(E, \alpha | E', \alpha') = u_-(E, \alpha | E', \alpha') . \tag{9.11}$$

We shall then have, as follows from (7.28) and (7.29), the result that $f_0(E, \alpha; t)$ and $\exp[-iFt]f_-(F, \beta)$ are the H_0 and H representatives, respectively, of the solution of $\Phi(t)$ of the Schrödinger equation

$$H\Phi(t) = i \frac{\partial \Phi(t)}{\partial t} , \tag{9.12}$$

with the initial condition

$$\lim_{t \to -\infty} \exp[iH_0 t]\Phi(t) = \Phi_- \underset{E}{\longleftrightarrow} f_-(E, \alpha) .$$

This statement is essentially the adiabatic theorem for the continuous spectrum which can be worded as follows: When a perturbation is switched on infinitely slowly, a superposition of eigenstates of the unperturbed Hamiltonian in the infinite past goes over into the *identical* superposition of corresponding eigenstates of the total Hamiltonian at finite times.

In our discussion it will be convenient to introduce another integral operator with the kernel $v_{\zeta,t}(E, \alpha | E', \alpha')$ defined by

$$v_{\zeta,t}(E, \alpha | E', \alpha') = w_{\zeta,t}(E, \alpha | E', \alpha') \exp[i(E - E')t] . \tag{9.13}$$

Hence

$$\exp[iEt] f_\zeta(E, \alpha; t) = \int_{E_a}^{E_b} \int v_{\zeta,t}(E, \alpha | E', \alpha') f_-(E', \alpha') \mathrm{d}\alpha' \, \mathrm{d}E'. \tag{9.14}$$

The initial condition (9.09) can now be interpreted as an initial condition on $v_{\zeta,t}(E, \alpha | E', \alpha')$, namely,

$$\lim_{t \to -\infty} v_{\zeta,t}(E, \alpha | E', \alpha') = \delta(E - E') \, \delta(\alpha, \alpha') . \tag{9.15}$$

The simplicity of the initial condition on $v_{\zeta,t}$ motivates its introduction in the present context. It should be pointed out that the kernel or matrix $v_{\zeta,t}(E, \alpha | E', \alpha')$ is just the operator U_+ of LIPPMANN and SCHWINGER as expressed in the H_0 representation. This operator also appear in FUBINI's paper.

Substitution of (9.14) into the differential equation (9.08) for $f_\zeta(E, \alpha; t)$ and the use of the fact that $f_-(E, \alpha)$ is largely arbitrary leads us to the fol lowing equation for $v_{\zeta,t}$:

$$\frac{\partial}{\partial t} v_{\zeta,t}(E, \alpha | E', \alpha') = -i\varepsilon \int_{E_a}^{E_b}\int V^E(E, \alpha | E'', \alpha'') \exp\left[[i(E - E'') + \zeta]t\right] \cdot \cdot v_{\zeta,t}(E'', \alpha'' | E', \alpha') \mathrm{d}\alpha'' \mathrm{d}E'', \tag{9.16}$$

or on integration with respect to t and on using the initial condition (9.15) we find

$$v_{\zeta,t}(E, \alpha\ E', \alpha') = \delta(E - E')\,\delta(\alpha, \alpha') - i\varepsilon \int_{-\infty}^{t} \mathrm{d}t' \int_{E_a}^{E_b}\int V^E(E, \alpha | E'', \alpha'') \cdot \cdot \exp\left[[i(E - E'') + \zeta]t'\right] v_{\zeta,t'}(E'', \alpha'' | E', \alpha') \mathrm{d}\alpha'' \mathrm{d}E'' . \tag{9.17}$$

Having used our initial condition, we express $v_{\zeta,t}$ in terms of $w_{\zeta,t}$((9.13)) and obtain an equation for $w_{\zeta,t}$

$$w_{\zeta,t}(E, \alpha | E', \alpha') = \delta(E - E')\,\delta(\alpha, \alpha') - i\varepsilon \exp\left[-i(E - E')t\right] \cdot \cdot \int_{-\infty}^{t} \mathrm{d}t' \int_{E_a}^{E_b}\int V^E(E, \alpha | E'', \alpha'') \exp\left[[i(E - E') + \zeta]t'\right] \cdot w_{\zeta,t'}(E'', \alpha'' | E', \alpha') \mathrm{d}\alpha'' \mathrm{d}E''. \tag{9.18}$$

In equation (9.18) let us now integrate the time integral by parts. We obtain

$$w_{\zeta,t}(E, \alpha | E', \alpha') = \delta(E - E')\,\delta(\alpha, \alpha') - \frac{i\varepsilon \exp[\zeta t]}{i(E - E') + \zeta} \cdot \cdot \int_{E_a}^{E_b}\int V^E(E, \alpha | E'', \alpha'') w_{\zeta,t}(E'', \alpha'' | E', \alpha') \mathrm{d}\alpha'' \mathrm{d}E'' + + \frac{i\varepsilon \exp[-i(E - E')t]}{i(E - E') + \zeta} \int_{-\infty}^{t} \mathrm{d}t' \int_{E_a}^{E_b}\int V^E(E, \alpha | E'', \alpha'') \exp\left[[i(E - E') + \zeta]t'\right] \cdot \cdot \frac{\partial}{\partial t'} w_{\zeta,t'}(E'', \alpha'' | E', \alpha') \mathrm{d}\alpha'' \mathrm{d}E'' . \tag{9.19}$$

Taking the limit as ζ approaches zero, we find

$$(9.20)\qquad w_{0,t}(E,\alpha\,|\,E',\alpha') = \delta(E-E')\delta(\alpha,\alpha') + \gamma_-(E'-E)\int_{E_a}^{E_b}\!\!\int V^E(E,\alpha\,|\,E'',\alpha'')\cdot$$

$$\cdot w_{0,t}(E'',\alpha''\,|\,E',\alpha')\,\mathrm{d}\alpha''\,\mathrm{d}E'' + i\varepsilon\exp\left[-i(E-E')t\right]\lim_{\zeta\to 0}\frac{1}{i(E-E')+\zeta}\cdot$$

$$\cdot\int_{-\infty}^{t}\mathrm{d}t'\int_{E_a}^{E_b}\!\!\int V^E(E,\alpha\,|\,E'',\alpha'')\exp\left[[i(E-E')+\zeta]t'\right]\frac{\partial}{\partial t'}\,w_{\zeta,t'}(E'',\alpha''\,|\,E',\alpha')\,\mathrm{d}\alpha''\,\mathrm{d}E''\,.$$

We see that if

$$(9.21)\qquad \lim_{\zeta\to 0}\frac{1}{i(E-E')+\zeta}\int_{-\infty}^{t}\mathrm{d}t'\int_{E_a}^{E_b}\!\!\int V^E(E,\alpha\,|\,E'',\alpha'')\exp\left[[i(E-E')+\zeta]t'\right]\cdot$$

$$\cdot\frac{\partial}{\partial t'}\,w_{\zeta,t'}(E'',\alpha''\,|\,E',\alpha')\,\mathrm{d}\alpha''\,\mathrm{d}E'' = 0\,,$$

then $w_{0,t}$ satisfies the same integral equation as the eigenfunction u_-. Since the solution of this integral equation can be shown to be unique, we have the result that (9.21) is a necessary and sufficient condition for $w_{0,t}$ to be identical to the eigenfunction u_-. That is, relation (9.21) is a necessary and sufficient condition for the equivalence of the adiabatic switch-off procedure and the use of the spectral representation of H in the time-dependent formalism in scattering problem. In our treatment (and that of references [3] and [12]) no explicit conditions have been put on the total Hamiltonian H. However, condition (9.21) is essentially a condition on the perturbation and hence also on H.

A sufficient condition for the validity of the adiabatic theorem can be seen to be

$$(9.22)\qquad \lim_{\zeta\to 0}\frac{\partial}{\partial t}\,w_{\zeta,t}(E,\alpha\,|\,E',\alpha') = 0\,,$$

since in this case (9.21) is satisfied. We shall now give arguments to indicate that (9.22) and hence the adiabatic theorem is valid in many cases, even when H has point eigenvalues, a case not included in Snyder's or Friedrich's treatments.

From (9.19) it can be shown that $\partial w_{\zeta,t}/\partial t$ satisfies the following equation.

$$(9.23)\quad \frac{\partial}{\partial t} w_{\zeta,t}(E,\alpha|E',\alpha') = \frac{\varepsilon}{(E'-E)+i\zeta}\Big\{\zeta \exp[\zeta t]\int_{E_a}^{E_b}\int V^E(E,\alpha|E'',\alpha'')\cdot$$
$$\cdot w_{\zeta,t}(E'',\alpha''|E',\alpha')\,\mathrm{d}\alpha''\,\mathrm{d}E'' + i(E-E')\exp[-i(E-E')t]\cdot$$
$$\cdot\int_{-\infty}^{t}\mathrm{d}t'\int_{E_a}^{E_b}\int V^E(E,\alpha|E'',\alpha'')\cdot$$
$$\cdot\frac{\partial}{\partial t'} w_{\zeta,t'}(E'',\alpha''|E',\alpha')\exp\big[[i(E-E')+\zeta]t'\big]\,\mathrm{d}\alpha''\,\mathrm{d}E''\Big\}.$$

We note that that we can write

$$w_{\zeta,t}(E,\alpha|E',\alpha') = \int_{-\infty}^{t}\mathrm{d}t'\,\frac{\partial}{\partial t'} w_{\zeta,t'}(E,\alpha|E',\alpha') + w_{\zeta,-\infty}(E,\alpha|E',\alpha')\,.$$

However, from (9.19) it is clear that

$$w_{\zeta,-\infty}(E,\alpha|E',\alpha') = \delta(E-E')\,\delta(\alpha,\alpha')\,,$$

so that (9.23) becomes

$$(9.24)\quad \frac{\partial}{\partial t} w_{\zeta,t}(E,\alpha|E',\alpha') = \frac{\varepsilon\exp[\zeta t]}{(E'-E)+i\zeta}\Big\{\zeta V^E(E,\alpha|E',\alpha') +$$
$$+\int_{-\infty}^{t}\mathrm{d}t'\int_{E_a}^{E_b}\int V^E(E,\alpha|E'',\alpha'')\big\{\zeta + i(E-E')\exp[i(E-E')+\zeta](t'-t)\big\}\cdot$$
$$\cdot\frac{\partial}{\partial t'} w_{\zeta,t'}(E'',\alpha''|E',\alpha')\,\mathrm{d}\alpha''\,\mathrm{d}E''\Big\}.$$

We can solve equation (9.24) by iteration. We find that

$$(9.25)\quad \frac{\partial}{\partial t} w_{\zeta,t}(E,\alpha|E',\alpha') = \zeta\Big[\varepsilon\exp[\zeta t](E'-E+i\zeta)^{-1}V^E(E,\alpha|E',\alpha') +$$
$$+2\varepsilon^2\exp[2\zeta t](E'-E+2i\zeta)^{-1}\int_{E_a}^{E_b}\int V^E(E,\alpha|E'',\alpha'')(E'-E''+i\zeta)^{-1}\cdot$$
$$\cdot V^E(E'',\alpha''|E',\alpha')\,\mathrm{d}\alpha''\,\mathrm{d}E'' +$$
$$+3\varepsilon^3\exp[3\zeta t](E'-E+3i\zeta)^{-1}\int_{E_a}^{E_b}\int_{E_a}^{E_b}\int_{E_a}^{E_b} V^E(E,\alpha|E'',\alpha'')(E'''-E''+i\zeta)^{-1}\cdot$$
$$\cdot V^E(E'',\alpha''|E''',\alpha''')(E'-E'''+2i\zeta)^{-1}V^E(E''',\alpha'''|E',\alpha')\cdot\mathrm{d}\alpha''\,\mathrm{d}E''\,\mathrm{d}\alpha'''\,\mathrm{d}E''' + \ldots\Big].$$

If the series with the square brackets converges as ζ approaches zero, then it is clear that (9.22) is satisfied. Now when $\zeta \to 0$ the series in brackets is within a factor ε the derivative with respect to ε of the series for u_- obtained by iteration from (7.19*a*). Hence if the series for u_- converges, i.e., if u_- is an analytic function in ε, the series in brackets will converge, since the derivative of an analytic function is also analytic and hence is representable as a power series with the same radius of convergence. We thus come to the conclusion that a sufficient condition for the validity of the adiabatic theorem is the convergence of the power series in ε for u_-. Since there are examples in which the series for u_- converges even when H has point eigenvalues (see e.g., ref. ([5])) we have indicated that there are cases even when the spectrum of H differs from that of H_0 where the adiabatic theorem holds.

We could have obtained the above results by solving equation (9.18) directly by iteration and then allowing ζ to approach zero. We would then have found that the series expression for $w_{0,t}$ is identical to that for u_-. This procedure is, in fact, used by SNYDER in his discussion of the «irrelevance of the adiabatic theorem». However, it appears useful to indicate explicit conditions, namely, (9.21) or (9.22) under which the adiabatic theorem will hold.

It might be noted that the convergence of the power series for u_- is a sufficient condition only, so that it is possible that the adiabatic theorem will hold even when such a series expansion is not valid. This appears to be the main point of SNYDER's treatment in which he assumes $H_\zeta(t)$ has the same spectrum of H_0 for all values of ζ and t.

RIASSUNTO (*)

Si dimostra che le autofunzioni di entrata e di uscita che si impiegano nella teoria dello scattering si possono ottenere specificando una relazione fra gli stati iniziali o finali e lo stato al tempo $t=0$. Così diviene superfluo il far ricorso al teorema adiabatico.

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On Some General Properties of Static Solutions of Schiff's Equation.

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Summary. — Some general properties of the static solutions of Schiff's equation are derived in Section 1 from the structure of the non-linear equation and the behavior of the source distribution. Section 2 is devoted to the proof of the existence of the solution, and to a derivation of the many-body forces implied in the classical scalar theory of Schiff potentials.

Introduction.

In an attempt to account for both saturation and the independent-nucleon model of atomic nuclei, Schiff (1) has described the interactions between nucleons as arising from mesons which obey a non-linear wave equation. We are led to the same equation as a result of the renormalization process. Now many difficulties arise in dealing with the corresponding quantum field theory which have not yet been solved. On the other hand, it is clear by now that other terms than the Φ^3 term must also play a role in explaining saturation. For both reasons we shall consider Schiff's equation from the phenomenological standpoint and treat the non-linear field as a classical field.

In the present paper we shall deal with the nature and some general properties of the static solutions of Schiff's equation which can be derived in a rather simple way from the structure of both the non-linear equation and source distribution. Most of the results we shall arrive at are quite insensitive

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(1) J. L. Schiff: *Phys. Rev.*, **84**, 1 and 10 (1951); **86**, 856 (1952); **92**, 766 (1953).

to the particular analytical form of the source distribution, being mere mathematical consequences of the type of non-linearity exhibited by the partial differential equation. The assumptions we shall accordingly make as to the behavior of the source distribution will be large enough to include all cases of physical interest, with the only exception of point sources. Though in the theory appear two parameters, namely the coupling constant g and the coefficient α^2 of the Φ^3 term, only the product $\lambda \equiv \alpha g$ of the two plays an essential role in determining the behavior of the field for a given nucleon distribution. The field thus appears as a function of λ and a functional of the nucleon distribution $f(x)$, being everywhere proportional to the coupling constant for given λ and $f(x)$. In Section **1** we define the mathematical problem and examine the general properties of the static solutions of Schiff's equation assuming that such solutions do exist. Section **2** is devoted to the proof of the existence of the solution, and to a derivation of the many-body forces among nucleons implied in the classical scalar theory of Schiff potentials.

1. – Behavior of the Static Solution of Schiff's Equation.

1·1. – In this first paper on non-linear meson theory of nuclear forces, Schiff describes the interactions between nucleons as arising from mesons which obey the non-linear wave equation

$$\frac{\partial^2 \Phi}{\partial t^2} = \Delta\Phi - \Phi - \alpha^2\Phi^3 + gf\,, \tag{1}$$

where Φ is the meson field amplitude, f the nucleon distribution, g the coupling constant and α^2 a constant. Units are chosen in such a way that c, $\hbar$, and the meson mass are all equal to unity.

We shall assume that the distribution $f(x)$ does not depend on time and that $f(x)$ tends to zero u.d. (that is, uniformly with regard to the direction) as $r \equiv |x|$ tends to infinity. However, most of the properties can be extended to the case in which $f(x) \to f_N$ u.d. as $r \to \infty$, where f_N is a non zero constant. This is the situation that would result from the superposition of a more or less fluctuating nuclear distribution, which is zero at infinity, and a constant « nucleon background » f_N. To put it still another way, we are faced with a distribution of this type when one or more nucleons are embedded in a sea of nuclear matter represented by the constant density f_N, as opposed to the case in which the nucleon or nucleons are considered in empty space.

In this paper we shall mainly be concerned, however, with source distributions that are zero at infinite distance, and only incidentally the possible extensions to the more general case will be pointed to. We shall in particular

discuss nucleon distributions of the form

$$f(x) = \sum_{i=1}^{A} \varrho(x - x_i) , \tag{2}$$

where $\varrho(x - x_i)$ represents the nucleon density for *one* nucleon. It will be assumed that $\varrho(x - x_i)$ is a positive monotonic decreasing function of the distance $|x - x_i|$ from the fixed point x_i to the variable point x, and that $\varrho(x - x_i)$ goes to zero when $|x - x_i| \to \infty$ in such a way that it is practically zero for $|x - x_i| > a$, where a is a small quantity. We will then speak of a nucleon *at* x_i and radius a. We thus consider the nucleon as a source of finite extent But it is well known that the theory runs into difficulties when attempting to deal with point sources, i.e., when the function ϱ behaves as a δ-function. Further, we shall assume the function $\varrho(x - x_i)$ to be continuous throughout the space. This assumption excludes sources of the type $\varrho(x - x_i) = 0$ for $|x - x_i| > a$, $\varrho(x - x_i) \neq 0$ for $|x - x_i| \leqslant a$, i.e. the case of nucleons of sharply defined extent. This hypothesis does not entail any essential loss of generality from the physical point of view, since we can always smooth out any discontinuous dropping to zero of the density at $|x - x_i| = a$. As a matter of fact, the assumption of a continuous nucleon distribution $f(x)$ simplifies the reasonings without severely restricting the theory. All the results we shall arrive at are also true when the source distribution $f(x)$ is discontinuous, with *finite* discontinuities, on a finite number of closed regular surfaces, thus in particular for a finite assembly of sharply defined nucleons. We shall also admit that $\int \varrho(x - x_i)\,dx$ is finite, and will normalize $\varrho(x - x_i)$ in such a way that $\int \varrho(x - x_i)\,dx = 1$. All the nucleons of an assembly of A nucleons will be considered as equivalent, and so the functions $\varrho(x - x_i)$ in Eq. (2) will differ only in the values of the parameter x_i. Under this conditions and for nucleon distributions of the form Eq. (2), the coupling constant g in Eq. (1) is a well defined quantity, and $\int f(x)\,dx = A$. Finally, we shall find it convenient to change the notation by writing

$$V(x) \equiv \frac{1}{g} \Phi(x) . \tag{3}$$

The boundary problem of which $V(x)$ must be the solution takes then the form (see Eq. (1))

$$\text{(P)} \qquad \begin{cases} \Delta V - V - \lambda^2 V^3 + f(x) = 0 , \\ V(x) \to 0 \text{ u.d. as } |x| \to \infty , \end{cases} \tag{4}$$

where

$$\lambda \equiv \alpha g . \tag{5}$$

The value of $V(x)$ at a point x depends on all the values of $f(x)$ and on the particular value of the parameter λ. In the following, we shall accordingly deal with some simple properties of $V(x)$ considered as a functional of the nucleon distribution and as a function of the parameter λ.

By a solution of problem (P) we shall always mean a function $V(x)$ that satisfies the partial differential equation (4) and the boundary condition at infinity, the function $V(x)$ itself being continuous together with its partial derivatives of the first two orders throughout the space. However, when $f(x)$ exhibits discontinuities of the type referred to above, the laplacian ΔV will also be discontinuous on the same surfaces and with the same discontinuities as the source distribution itself, though of opposite sign. The function $f(x)$ will not necessarily be everywhere one-signed, but if it is so, it will be assumed to be positive — or at least non negative — throughout the space. The relations obtained in the case $f(x) \geqslant 0$ are, in fact, readily translated into those holding in the case $f(x) \leqslant 0$.

1·2. – We proceed now to formulate without proof two theorems which will be extremely useful for the subsequent analysis. Simple cases of them appear obvious from physical intuitions, and formal proofs can be found in any standard book on partial differential equations. Some care must be taken, however, when they are proved under the assumption that the region of space involved is bounded, since this is not the case here.

We formulate the first theorem as follows:

Theorem 1. Let $\varphi(x)$ be a *positive* continuous function of x throughout the space. The boundary problem

$$(6) \qquad (P_1) \qquad \begin{cases} \Delta V - \varphi(x) V = 0 , \\ V(x) \to 0 \text{ u.d. as } |x| \to \infty , \end{cases}$$

has no other solution, continuous with continuous partial derivatives of the first two orders, than the $V(x) \equiv 0$.

From this theorem we can immediately deduce two corollaries of great importance for our problem. First observe that, in the absence of sources ($f(x) \equiv 0$), the only solution (cfr. § 1·1) of problem P, Eq. (4), is $V(x) = 0$. For any solution of problem (P) with $f(x) \equiv 0$ is a solution of problem (P_1) with

$$\varphi(x) = 1 + \lambda^2 V(x)^2 \geqslant 1$$

and hence $V(x) = 0$. As pointed out by Schiff, we may interpret this result by saying that mesons cannot be permanently localized when there is no source. The same lack of permanent locability in the absence of sources would

be true for *any* meson field governed by a wave equation of the form

$$\frac{\partial^2 \Phi}{\partial t^2} = \Delta\Phi - \Gamma(x, \Phi)\Phi + gf,$$

where $\Gamma(x, \Phi)$ is any *positive* continuous function of x and Φ for all values of x and Φ. On the other hand, it is by now clear that the nucleon distribution determines uniquely the meson field, since if there were two fields V_1 and V_2 associate to the same distribution $f(x)$, the difference $V_1 - V_2$ would be a solution of problem (P_1) with

$$\varphi(x) \equiv 1 + \lambda^2(V_1^2 + V_1 V_2 + V_2^2) \geqslant 1$$

and hence necessarily $V_1 - V_2 \equiv 0$. In other words, if we get a solution of problem (P), this is *the* solution and defines the field caused by the given source distribution.

The second theorem refers to some inequalities ... boundary problem

$$(7) \qquad (P_2) \qquad \begin{cases} \Delta V - \varphi(x)V + f(x) = 0, \\ V(x) \to 0 \text{ u.d. as } |x| \to \infty, \end{cases}$$

where $f(x)$ and $\varphi(x)$ are given. Both are assumed to be continuous for all values of x, and $\varphi(x) > 0$ everywhere. The source distribution $f(x)$ can take positive and negative values, but if it is one-signed, it will be assumed that $f(x) \geqslant 0$ throughout the space.

Now: *Theorem 2* states that:

1) if $V(x)$ is the solution of problem (P_2), there are only two possible alternatives, namely

$$9a) \qquad \sup_x V(x) > 0, \qquad \inf_x V(x) < 0;$$

$$9b) \qquad \sup_x V(x) > 0, \qquad \inf_x V(x) = 0;$$

2) depending on which of the two alternatives above holds, we shall necessarily have one of the two following inequalities

$$(9a') \qquad \frac{f(x_m)}{\varphi(x_m)} \leqslant V(x) \leqslant \frac{f(x_M)}{\varphi(x_M)}, \qquad \text{with } f(x_m) < 0 \text{ and } f(x_M) > 0;$$

$$(9b') \qquad 0 \leqslant V(x) \leqslant \frac{f(x_M)}{\varphi(x_M)},$$

these inequalities holding for any value of x.

Here we mean by $\sup_x V (\inf_x V)$ the upper bound (the lower bound) of $V(x)$ throughout the space, and $x_M(x_m)$ is any point at a *finite* distance at which the field $V(x)$ takes its true maximum $\max_x V(x)$ (its true minimum $\min_x V(x)$).

Observe that in case a) both x_M and x_m exist, but that case b) does not ensure the existence of the true minimum. All these results are also true when $f(x)$ is discontinuous, with finite discontinuities, on a finite number of regular closed surfaces.

From the theorem above it immediately follows that, if $f(x) \geqslant 0$ everywhere, then we shall also have $V(x) \geqslant 0$ for all values of x, though the converse is not in general true. Under the same hypothesis $f(x) \geqslant 0$, $V(x)$ can be zero only at those points at which $f(x) = 0$, too, so that $f(x) > 0$, all x, entails $V(x) > 0$ everywhere.

Another fairly obvious consequence of *Theor. 2* is that, *if* $\varphi(x) \geqslant 1$, then the fields V_1 and V_2 determined by the source distributions f_1 and f_2, with $|f_1(x) - f_2(x)| < k$ throughout the space, are such that $|V_1(x) - V_2(x)| < k$ for all values of x. This follows from the fact that the difference $V_1 - V_2$ is the solution of problem (P_2) for $f \equiv f_1 - f_2$ as source distribution and from relations $(9a')$ and $(9b')$. The physical interpretation of this result is rather trivial. It states that, if two source distributions differ *uniformly* in less than, say, a very small quantity ε, the associate fields will also differ in less than *the same* quantity throughout the space.

Finally, it is easily seen that, if $f(x) \geqslant 0$, then *increasing* $\varphi(x)$ everywhere *decreases* the potential $V(x)$ throughout the space, with the only possible exception of those points at which $V(x)$ cannot decrease remaining non-negative, i.e., the points at which $V(x) = 0$. In particular, the static Neumann-Yukawa potential [2]

$$V_0(x) \equiv \int G(x - x_0) f(x_0) \, \mathrm{d}x_0 ,$$

with $G(x - x_0) = \exp[-|x - x_0|]/4\pi|x - x_0|$ and $f(x_0) \geqslant 0$, is everywhere positive and will be greater than any static potential that obeys a field equation of the form

$$\Delta V - \Gamma(x, V) V + f = 0 ,$$

where $\Gamma(x, V) > 1$ for all values of x and V. For example, in the case of Schiff potentials we have $\Gamma(x, V) \equiv 1 + \lambda^2 V(x)^2 > 1$ with the only possible exception of the points at which $V(x) = 0$. But at these points $V_0(x) > 0$, Hence, if $f(x) \geqslant 0$, then $V(x) < V_0(x)$ everywhere. In other words, given a static non-negative nucleon distribution and if mesons obey Schiff non-linear equation, the field amplitude must be everywhere less than the amplitude

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[2] These potentials were considered extensively by C. NEUMANN: *Allgemeine Untersuchungen über das Newtonische Prinzip der Fernwirkungen mit besonderer Rücksicht auf die elektrischen Wirkungen* (Teubner, 1896).

of the classical Yukawa field for the *same* nucleon distribution and meson mass. More precisely, if g and g_0 are the coupling constants for the Schiff and Yukawa fields, respectively, we shall have for all values of x [see Eq. (3)]

$$\Phi(x) < \frac{g}{g_0}\Phi_0(x)\,.$$

We shall come back to this point in § 1·5.

1·3. – We shall now apply all these results to the non-linear meson field $V(x)$ that obeys Eq. (4). We have already seen that problem (P) has either only one solution or no solution at all. To put it still another way, given the nucleon distribution $f(x)$ and the parameter λ, either we shall have a well determined field $V(x)$, or the nucleon distribution and/or the value of λ are not compatible with the theory. For the time being, we shall therefore assume that *the* solution of problem (P) exists for all $f(x)$ and λ which will come in play.

The essential point is that the solution $V(x)$ of problem (P) is also the solution of problem (P_2) with the same source and $\varphi(x) = 1 + \lambda^2 V(x)^2 \geqslant 1$. But now we can be a little more precise than before, since we know the form of the function $\varphi(x)$. For example, since $V(x)$ and $\varphi(x)V(x) = V(x) + \lambda^2 V(x)^3$ are now maximum or minimum *at the same points*, we can write, instead of (9a') and (9b'),

$$(10a) \qquad f(x_m) \leqslant V(x) + \lambda^2 V(x)^3 \leqslant f(x_M), \qquad \text{with } f(x_m) < 0 \text{ and } f(x_M) > 0\,,$$

$$(10b) \qquad 0 \leqslant V(x) + \lambda^2 V(x)^3 \leqslant f(x_M)\,,$$

respectively. The first relation holds when $V(x)$ is not one-signed, the second when $V(x) \geqslant 0$. As in *Theor. 2*, $x_M(x_m)$ is any point at which the field takes its true maximum (its true minimum). We see then that the static field described by Eq. (4) is everywhere less than (greater than) the value of the nucleon density at the point at which *the field* exhibits its true maximum (its true minimum) and *a fortiori* less than (greater than) the upper bound F (lower bound f) of the nucleon density, which *does not depend* on λ. If the nucleon distribution is one-signed, the field is also one-signed and both have the same sign.

Now, from (10a) it follows that

$$(11a) \qquad 0 \leqslant |V(x)|^3 < \frac{1}{\lambda^2}\sup_x |f(x)|\,, \qquad \text{all } x\,,$$

and from (10b)

$$(11b) \qquad 0 \leqslant V(x)^3 < \frac{F}{\lambda^2}\,, \qquad \text{all } x\,.$$

Hence a positive constant K, independent of λ, always exists such that $|V(x)| < K\lambda^{-\frac{2}{3}}$ for *all* values of x and λ. In other words, the field $V(x)$ tends to zero as $\lambda \to \infty$ in such a way that

$$V(x) = 0(\lambda^{-\frac{2}{3}}), \tag{12}$$

uniformly throughout the space as $\lambda \to \infty$. This result has a simple, though important, physical interpretation. Given a *static* nucleon distribution $f(x)$ and the value of the coupling constant g, we cannot enhance the non-linearity of the theory by increasing the constant α^2 beyond any limit, since then either there is no solution, or the field will become vanishingly small tending everywhere to zero not less rapidly than $\alpha^{-\frac{2}{3}}$. For example, in the case $f(x) \geqslant 0$ we shall have, by Eqs. (12) and (5),

$$0 \leqslant \Phi(x, \lambda) < \frac{(gF)^{\frac{1}{3}}}{\alpha^{\frac{2}{3}}}. \tag{13}$$

Consider now the fields $V_1(x)$ and $V_2(x)$ determined by the nucleon distributions $f_1(x)$ and $f_2(x)$, respectively, and suppose that $f_1(x) > f_2(x)$ throughout a region R, $f_1(x)$ being equal to $f_2(x)$ at all points which do not belong to R. Then by considering as before the difference $V_1 - V_2$ as the solution of problem (P_2) with the source $f_1 - f_2$ and $\varphi(x) = 1 + \lambda^2(V_1^2 + V_1V_2 + V_2^2) \geqslant 1$, it is easily seen from what was said in § 1·2 that *at least throughout* R is $V_1(x) > V_2(x)$ and that we shall have for all values of x

$$0 \leqslant V_1(x) - V_2(x) \leqslant \left(\frac{f_1(x) - f_2(x)}{1 + \lambda^2(V_1^2 + V_1V_2 + V_2^2)}\right)_{x = x_M} < \sup_x (f_1(x) - f_2(x)),$$

where x_M is a point at which $V_1 - V_2$ takes its true maximum. That is, if the nucleon density is increased throughout a region R, the field amplitude increases *at least* everywhere throughout R, but the increment of the field at *any* point is always less than the maximum increase of the nucleon density.

We shall next consider the effect of superposing two «absolutely rigid» nucleon distributions $f_1(x)$ and $f_2(x)$, both non-negative. We are thus ignoring the possible correlational changes of $f_1(x)$ and $f_2(x)$ brought about by the superposition. Let $V_1(x)$ and $V_2(x)$ be the fields determined by $f_1(x)$ and $f_2(x)$, respectively, and let U be defined by the identity

$$U \equiv V_{12} - (V_1 + V_2),$$

where V_{12} is the field determined by the source $f_1(x) + f_2(x) \geqslant 0$. It is easily seen from the partial differential equations and the common boundary con-

dition for V_1, V_2 and V_{12}, that U is a solution of problem (P_2) with

$$\varphi(x) \equiv 1 + \lambda^2\{(U + \tfrac{3}{2}(V_1 + V_2))^2 + \tfrac{3}{4}(V_1 + V_2)^2\} \geqslant 1\,,$$
$$f(x) \equiv -3\lambda^2(V_1^2 V_2 + V_1 V_2^2)\,.$$

But $f_1(x) \geqslant 0$ and $f_2(x) \geqslant 0$. Hence $V_1(x) \geqslant 0$ and $V_2(x) \geqslant 0$, and so $f(x) \leqslant 0$, i.e., $U(x) \leqslant 0$ for all values of x. Further, $U(x)$ can only be zero at the points at which at least one of the two potentials $V_1(x)$ and $V_2(x)$ is zero. Therefore, with the only possible exceptions of those points [see § 1·5], we shall have

$$V_{12} < V_1 + V_2$$

throughout the space if $\lambda > 0$. As was to be expected, U vanishes identically when $\lambda = 0$, i.e., in the linear case and *only* in this case. On the other hand, we know that $V_{12} > V_1$ and $V_{12} > V_2$. Hence, we can assert that

$$\tfrac{1}{2}(V_1 + V_2) < V_{12} < V_1 + V_2 \tag{14}$$

and that $V_{12} \equiv V_1 + V_2$ if and only if $\lambda = 0$.

1·4. – It might be well to discuss here briefly the more general case $f \to f_N \neq 0$ u.d. as $|x| \to \infty$, with f_N a constant. When the nucleon distribution does not g_0 to zero at infinity, but tends to a constant, say, positive value f_N, the analysis becomes somewhat more complicated. However, the general method of approach is the same as that already discussed.

We first introduce the functions U and h defined by

$$U \equiv V - V_N\,, \qquad h \equiv f - f_N\,, \tag{15}$$

where V_N is the only real root of the equation $V_N + \lambda^2 V_N^3 = f_N$ and V the Schiff field determined by the given nucleon distribution f with the boundary condition $V \to V_N$ u.d. as $|x| \to \infty$. Clearly, U must be solution of the boundary problem

$$\Delta U - [1 + \tfrac{3}{4}\lambda^2 V_N^2 + \lambda^2(U + \tfrac{3}{2}V_N)^2]U + h = 0, \quad U \to 0 \text{ u.d. as } |x| \to \infty. \tag{16}$$

Comparison with problem (P_2) shows that, in the present case,

$$\varphi(x) \equiv 1 + \tfrac{3}{4}\lambda^2 V_N^2 + \lambda^2(U + \tfrac{3}{2}V_N)^2 \geqslant 1 + \tfrac{3}{4}\lambda^2 V_N > 1\,,$$

and we can apply all theorems above. For example, it follows immediately that, if $h(x) \geqslant 0$, then $U(x) \geqslant 0$. Further, since $U(x)$ and $U(x)\varphi(x)$ are maximum

(or minimum) at the same points, we shall have

$$0 \leqslant |U|[1 + \tfrac{3}{4}\lambda^2 V_N^2 + \lambda^2(U + \tfrac{3}{2}V_N)^2] < \sup_x |h(x)|, \tag{17}$$

as before [see Eqs. (10)], with $U(x) \geqslant 0$ if $h(x) \geqslant 0$. From the inequality (17) it follows that, for *all* values of x,

$$|U(x)| < \frac{1}{1 + \tfrac{3}{4}\lambda^2 V_N^2} \sup_x |h(x)|, \tag{18}$$

where, if we *a priori* know that $U \geqslant 0$ (as would be the case if $h \geqslant 0$), the first factor on the right could be susbstituted by $1/(1 + 3\lambda^2 V_N^2)$. We see, therefore, that the contribution of the supplementary nucleon distribution $h(x)$ to the total field becomes vanishingly small when the density of the background f_N, and therefore V_N, increases tending to infinity. Of course, this effect vanishes in the linear case, when U does not depend on V_N. It is also readily proved that, when $h(x) \geqslant 0$, the fields V_v and U caused by the *same* nucleon distribution h in empty space and in a sea of nuclear matter represented by the constant $f_N > 0$, respectively, are such that the field U in nuclear matter is everywhere less than the field in empty space V_v, both being non-negative. In a certain sense, the presence of a background tends to suppress the effect of any superimposed nucleon distribution zero at infinity.

Relation (18) suggests the possibility of approximately computing U when λ, V_N and h are such that

$$\frac{1}{1 + \tfrac{3}{4}\lambda^2 V_N^2} \sup_x |h(x)| \ll V_N. \tag{19}$$

For then we shall have, by Eq. (18), $|U(x)| \ll V_N$ throughout the space, and we can expect the solution of the *linear* partial differential equation

$$\Delta U - (1 + 3\lambda^2 V_N^2)U + h = 0,$$

which is zero at infinity to be a good approximation for the exact solution of Eq. (16), at least asymptotically, for very high values of λV_N. This means in physical terms that, as long as condition (19) holds, the supplementary field U behaves as the ordinary Yukawa field determined by the same nucleon distribution $h(x)$, *but with a reduced range* $\varkappa^{-1} = (1 + 3\lambda^2 V_N^2)^{-\frac{1}{2}}$ instead of unity.

On the other hand, the energy density of the *total* field $V = V_N + U$ is easily seen to be

$$\mathcal{H} = -\tfrac{1}{2}g^2(V_N^2 + \tfrac{3}{2}\lambda^2 V_N^4) - \tfrac{1}{2}g^2(2hV_N + hU + \lambda^2 V_N U^3 + \tfrac{1}{2}\lambda^2 U^4). \tag{20}$$

The first term on the right represents the contribution of the background of uniformly distributed nuclear matter. The second term represents the (exact) contribution of the nucleon fluctuation $h(x)$. The first component $-g^2V_N h(x)$ can be interpreted as arising from the interaction of the supplementary nucleons *with the background* and tends *to infinity* as $V_N \to +\infty$. The other three components represent either the interaction between supplementary nucleons, $-\frac{1}{2}g^2(hU + \frac{1}{2}\lambda^2U^4)$, or many-body interactions between these nucleons and the background, $-\frac{1}{2}g^2\lambda^2V_N U^3$. All of them tend *to zero* as $V_N \to +\infty$, if $\lambda \neq 0$. Only if $\lambda \neq 0$ is $\lim_{V_N\to\infty} hU = 0$. As was expected, the non-linearity ($\lambda \neq 0$) tends to suppress all these interactions for sufficiently high values of V_N, i.e., of f_N, and we are left with only the two-body interactions between supplementary nucleons and background $-g^2V_N h$. Now, if we take for $h(x)$ an expression of the form Eq. (2), the *total* contribution of the nucleon distribution $h(x)$ to the energy of the field is *asymptotically* independent of the « positions » x_i of the nucleons and equal to $-g^2V_N A$, where A is the number of supplementary nucleons. In other words, nucleons embedded in a sea of nuclear matter of sufficiently high uniform density do not practically interact between them, behave as if they were absolutely free, and each contributes to the energy of the field with a constant amount $-g^2V_N$. This result is quite general. In particular it does not depend on the « size and shape » of the nucleon $\varrho(x - x_i)$ under the only condition of this function being everywhere finite. As a matter of fact V_N can be relatively small, and even then the main interaction can be that of the nucleons with the background. All then depends on the value of λ. Take, for example, the values given by SCHIFF [v. 1, p. 6], $g = 1.49$, $\lambda = \varkappa g = 11.86$, $V_N = 0.1$, and suppose the nucleons be represented by homogeneous, nonoverlaping spheres of radius $\varepsilon = 1$ in our units. The ratios of the second, third and fourth terms to the first in the expression for the supplementary energy density are then of order of magnitude or less than 23%, 2.8%, and 0.6%, respectively.

1·5. – We can now turn to the question as to how does the field $V(x)$, Eq. (4), behave considered as a function of the parameter λ, with $f(x)$ fixed. We shall indicate the explicit dependence of the field on λ by writing $V(x, \lambda)$ for $V(x)$. Suppose $V' \equiv V(x, \lambda')$ and $V'' \equiv V(x, \lambda'')$ are the solutions of problem (P) for two values λ' and $\lambda''(\lambda' < \lambda'')$ of the parameter λ and the same nucleon distribution. By considering $V' - V''$ as the solution of problem (P_2) with

$$\varphi(x) \equiv 1 + \lambda''^2(V'^2 + V'V'' + V''^2) \geqslant 1\,,$$
$$f(x) \equiv (\lambda''^2 - \lambda'^2)V'^3,$$

it is readily proved that, if V' is not one-signed,

$$(21)\quad (\lambda''^2 - \lambda'^2)\inf_x V(x, \lambda')^3 < V(x, \lambda') - V(x, \lambda'') < (\lambda''^2 - \lambda'^2)\sup_x V(x, \lambda')^3$$

and that, if $V' \geqslant 0$ everywhere,

$$0 \leqslant V(x, \lambda') - V(x, \lambda'') < (\lambda''^2 - \lambda'^2) \sup_x V(x, \lambda')^3. \tag{22}$$

From Eqs. (21), (22) and (11) it follows that: *a*) $V(x, \lambda)$ is a continuous function of λ throughout the space and the continuity is *uniform* with regard to x, in particular

$$\lim_{\lambda \to 0} V(x, \lambda) = V_0(x) \equiv \frac{1}{4\pi} \int \frac{\exp[-|x - x_0|]}{|x - x_0|} f(x_0)\, \mathrm{d}x_0; \tag{23}$$

b) if the derivative $\partial V(x, \lambda)/\partial\lambda$ exists, it will be everywhere finite, equal to zero for $\lambda = 0$, and will tend to zero as $\lambda \to +\infty$.

Consider now the particular case $f(x) \geqslant 0$. The Neumann-Yukawa potential $V_0(x)$, Eq. (23), is everywhere positive and, by Eq. (22), $V(x, \lambda)$ is a positive *decreasing monotonic* continuous function of λ [this ensures that the derivative $\partial V(x, \lambda)/\partial V$ will exist for almost all values of λ]. At each point x, $V(x, \lambda)$ reaches its maximum at $\lambda = 0$ and is equal there to the associate Neumann-Yukawa potential $V_0(x)$. Observe that from (22) it only follows that $\lambda' < \lambda''$ entails $V(x, \lambda') \geqslant V(x, \lambda'')$, i.e., that $V(x, \lambda)$ *cannot increase*. Now, the equality sign can hold only if $V(x, \lambda') = 0$, and then $V(x, \lambda) = 0$ for all $\lambda \geqslant \lambda'$. However, as we shall presently see, $V(x, \lambda)$ cannot be zero if $f(x) \geqslant 0$. Hence $V(x, \lambda)$ decreases monotonically and is everywhere positive.

Again, let $f(x)$ be everywhere non-negative. We know that $\lambda V(x, \lambda)$ is a continuous non-negative function of λ. Now, for two values $\lambda' < \lambda''$ of λ we shall have

$$\Delta(\lambda'' V'' - \lambda' V') - [1 + \lambda'^2 V'^2 + \lambda' \lambda'' V' V'' + \lambda''^2 V''^2](\lambda'' V'' - \lambda' V') + (\lambda'' - \lambda') f(x) = 0,$$

The bracket [ ] is $\geqslant 1$ and $(\lambda'' - \lambda') f(x) \geqslant 0$. Hence, $\lambda'' V'' - \lambda' V' \geqslant 0$, i.e., $\lambda V(x, \lambda)$ does not decrease as λ increases. But, for sufficiently small values of $\lambda > 0$, $\lambda V(x, \lambda)$ is positive. Hence $\lambda V(x, \lambda) > 0$ for *all* positive values, i.e., $V(x, \lambda)$ cannot be zero. On the other hand, it is easily proved that $\partial(\lambda V)/\partial\lambda = V + \lambda(\partial V/\partial\lambda)$ is positive for all values of λ, decreases monotonically with increasing λ, and tends to zero as $\lambda \to +\infty$. We can therefore assert that $\lambda V(x, \lambda)$ is a positive increasing monotonic function of λ. However, by Eq. (11*b*), $\lambda V(x, \lambda)$ cannot increase more rapidly than $(F\lambda)^{\frac{1}{2}}$, where F is the maximum value of the nucleon density $f(x)$. We thus have the situation qualitatively shown in fig. 1.

As a simple example, let us apply the results above to the field $\Phi(x, \lambda) = gV(x, \lambda)$. We still assume $f(x) \geqslant 0$. The value of $\lambda = \alpha g$ increases by increasing λ and/or g. Given the value of the coupling constant g, we now see that the field Φ is everywhere positive and decreases monotonically as α in-

creases, i.e., as the non-linearity is enhanced, tending to zero uniformly throughout the space as $\alpha \to +\infty$ [see Eq. (13)]. Consider now α fixed. We have

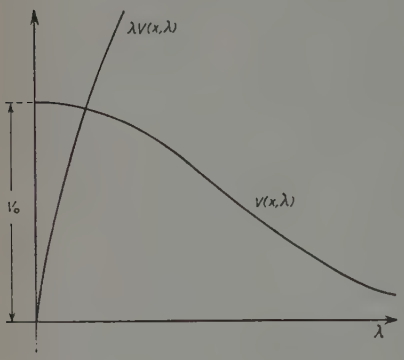

Fig. 1.

$$\Phi(x, \lambda) \equiv \frac{1}{\alpha} \cdot \lambda V(x, \lambda) .$$

As g increases, λ increases and $\lambda V(x, \lambda)$ increases monotonically. Hence, $\Phi(x)$ is an *increasing monotonic* function of the coupling constant g, though Φ can never increase more rapidly than $\alpha^{-\frac{2}{3}}(Fg)^{\frac{1}{3}}$. In the linear case, $\Phi(x)$ is everywhere proportional to g.

It may be interesting to note in this connection that there is a class of positive nucleon distributions $f(x)$ for which, *if* the solution of problem (P) exists for arbitrarily large values of λ, one has $\lim_{\lambda \to \infty} (V(x, \lambda) - f(x)/\lambda^{\frac{2}{3}}) = 0$ uniformly with regard to x, namely the class of positive functions $f(x)$ which tend to zero u.d. as $|x| \to \infty$ and for which

$$M \equiv \sup_x \left| f(x)^{-\frac{2}{3}} \Delta f - \tfrac{2}{3} f(x)^{-\frac{5}{3}} |\operatorname{grad} f|^2 - 3 f(x)^{\frac{1}{3}} \right| \tag{24}$$

exists and is finite. For example, in the simple case $f(x) = \exp[-r^2]$ the expression between the bars is equal to $(\frac{4}{3}r^2 - 9)\exp[-\frac{1}{3}r^2]$ and $M = 9$.

Assuming M exists and is finite for the given positive nucleon distribution $f(x)$, let $U(x, \lambda)$ be defined by

$$U(x, \lambda) \equiv \frac{f(x)}{\lambda^{\frac{3}{2}}} .$$

The function $U(x, \lambda)$ is the solution of problem (P) with the source distribution

$$f(x) - \frac{1}{3\lambda^{\frac{3}{2}}} \left[f(x)^{-\frac{2}{3}} \Delta f - \frac{2}{3} f(x)^{-\frac{5}{3}} |\operatorname{grad} f|^2 - 3 f(x)^{\frac{1}{3}} \right] .$$

Hence, according to what was said at the end of § 1·2, if $V(x, \lambda)$ is the field determined by the source $f(x)$, then

$$|V(x, \lambda) - U(x, \lambda)| < \frac{M}{3\lambda^{\frac{3}{2}}} ,$$

where the upper bound on the right does not depend on the point x. Therefore if $V(x, \lambda)$ exists for arbitrarily large values of λ and the given $f(x)$, for any

given $\varepsilon > 0$ there always exists λ_0, *independent* of x, such that $\lambda > \lambda_0$ entails

$$\left| V(x, \lambda) - \frac{f(x)^{\frac{1}{3}}}{\lambda^{\frac{1}{3}}} \right| < \varepsilon ,$$

throughout the space. In other words, if we increase the non-linearity by increasing α (i.e., λ), then the meson field, though everywhere small, will follow very closely the spacial behavior of the nucleon distribution, being smaller where f is small and larger where f is large: the field concentrates all around the sources [see also § 1·4].

2. – Non-linear Meson Theory and Many-Body Interactions.

2·1. – Before going to the proof of the existence of the solution of problem (P), it might be convenient to recall a well known theorem in the theory of the Neumann-Yukawa potential, namely the theorem that for every integrable function $f(x)$, and if $f(x) \to 0$ u.d. as $|x| \to \infty$, the potential

$$(25) \qquad V_0(x) \equiv \int G(x - x_0) f(x_0) \, \mathrm{d}x_0 \to 0 \qquad \text{u.d.} \qquad \text{as } |x| \to \infty ,$$

with $G(x - x_0) = \exp[-|x - x_0|]/4\pi|x - x_0|$. It is also well known that, under well defined conditions of continuity of $f(x)$ and its first partial derivatives, the function $V_0(x)$ defined in Eq. (25) is the only solution of problem (P) with $\lambda = 0$. We formulate next the following

Lemma. For every function $f(x)$, with $|f(x)|$ integrable, such that

$$(26) \qquad f(x) = O\left(\frac{e^{-r}}{r^{\beta}}\right) \quad \text{u.d.} \qquad (\beta > 2)$$

as $r \equiv |x| \to \infty$, we shall have

$$(27) \quad V_0(P) \equiv \int G(P - P_0) f(P_0) \mathrm{d}P_0 \sim \frac{e^{-R}}{4\pi R} \int \exp[\vec{OP}_0 \cdot \boldsymbol{n}] f(P_0) \, \mathrm{d}P_0 \quad \text{as } R \to \infty ,$$

where O is a fixed arbitrarily chosen point, $R \equiv |\vec{OP}|$ and $\boldsymbol{n} \equiv (1/R)\vec{OP}$(*). We get in this way a more precise information as to the asymptotic behavior of $V_0(x)$ than that furnished by the theorem Eq. (25). The asymptotic behavior (27) is nearly trivial when $f(x) = 0$ outside a *finite* region of space. But that

(*) This result holds *a fortiori* if $f(x) = 0\,(e^{-r}/r^{\beta})$, $\beta > 2$.

$V_0(x)$ will not in *general* behave as required by (27) is made clear by simply considering the source distribution $f(x) = 1/r$. The associate potential is then

$$V_0(R) = \frac{1 - e^{-2R}}{R} \sim \frac{1}{R} .$$

Further, since the integral on the right of the asymptotic Eq. (27) is not in general zero — certainly not when $f(x) \geqslant 0$ everywhere —, the principal part of the asymptotic expansion of $V_0(x)$ will *always* be of Yukawa type, irrespective of the analytical form of $f(x)$, as long as Eq. (26) is valid. The value of the integral on the right in Eq. (27) depends, of course, on the point O chosen and the direction $\boldsymbol{n}$ along which P tends to infinity. As a function of $\boldsymbol{n}$, the integral is a continuous function on the sphere of radius unity $|\boldsymbol{n}| = 1$, and so it is bounded. When the source distribution is spherically symmetric and we take O at the center of symmetry, Eq. (27) reduces to

$$\int G(x - x_0) f(x_0) \, \mathrm{d}x_0 \sim \frac{e^{-R}}{4\pi R} \cdot 4\pi \int_0^\infty f(r) r \operatorname{sh} r \, \mathrm{d}r .$$

A proof of this lemma is sketched in *Appendix I.*

2·2. – Let us turn back to the non-linear field $V(x)$ of problem (P). We know [see § **1·5**] that the potential $V(x)$ determined by a non-negative nucleon distribution $f(x) \geqslant 0$ is everywhere positive and bounded by the Neumann-Yukawa potential due to *the same* distribution. We have namely

$$0 < V(x) < V_0(x) \equiv \int G(x - x_0) f(x_0) \, \mathrm{d}x_0 . \tag{28}$$

This upper bound does not depend on λ. One can easily find a lower bound for $V(x)$, as can be seen in the following way. Consider the function

$$V_0(x, \lambda) \equiv V_0(x) - \lambda^2 \int G(x - x_0) V_0(x_0)^3 \, \mathrm{d}x_0 . \tag{29}$$

This function tends to zero u.d. ad $|x| \to \infty$ and is the solution of the boundary problem

$$\Delta V_1 - V_1 - \lambda^2 V_0(x)^3 + f(x) = 0 ,$$

$$V_1(x) \to 0 \quad \text{u.d. as} \quad |x| \to \infty .$$

The difference $V - V_1$, where V is the solution of problem (P), is the solution

of problem (P_2) with $\varphi(x) = 1$ and the source $\lambda^2(V_0^3 - V^3) > 0$ [see Eq. (28)]. Hence, for all values of x,

$$V_1(x) < V(x) . \tag{30}$$

Of course, $V_1(x)$ is of no use as a lower bound if it happens to be negative, since we know that $V(x)$ must be positive. Now, $V_1(x)$ can be negative at some points. In fact, given arbitrarily a point x we can always make $V_1(x) < 0$ at this point by taking λ sufficiently large. Hence the case $V_1(x) < 0$ cannot be excluded. However, let us proceed to investigate under which conditions can be $V_1(x) \geqslant 0$ for all values of x. We shall then have

$$0 \leqslant V_1(x) < V(x) < V_0(x) , \tag{31}$$

throughout the space. Now, it follows immediately from the definition Eq. (29) of $V_1(x, \lambda)$ that the necessary and sufficient condition that V_1 should be non-negative everywhere is that

$$\lambda^2 \leqslant \inf_x \frac{V_0(x)}{\int G(x - x_0) V_0(x_0)^3 \, \mathrm{d}x_0} = m . \tag{32}$$

It might be thought at first that the expression on the right will be zero, even when $V_0(x) > 0$ for all *finite* values of x, since $V_0(x) \to 0$ as $|x| \to \infty$. But there is a pretty large class of source distributions for which m is certainly not zero, namely that of all non-negative source distributions which fulfil the condition Eq. (26). We prove this statement in *Appendix II*. We thus have the result that, at least for a physically reasonable class of source distributions, $V_1(x)$ will be non-negative for sufficiently small values of λ, so that Eq. (31) will hold. But then the energy of the field V

$$H = -\frac{1}{2} g^2 \int \left(f V + \frac{1}{2} \lambda^2 V^4 \right) \mathrm{d}x \tag{33}$$

will be greater than the value of *the same integral* computed with the potential $V_0(x)$,

$$H_0 = -\frac{1}{2} g^2 \int \left(f V_0 + \frac{1}{2} \lambda^2 V_0^4 \right) \mathrm{d}x \tag{34}$$

and less than the integral

$$H_1 = -\frac{1}{2} g^2 \int \left(f V_1 + \frac{1}{2} \lambda^2 V_1^4 \right) \mathrm{d}x . \tag{35}$$

Note that the first statement is *always* true if $f(x) \geqslant 0$, but that, if $V_1(x)$ could

be negative, we could not infer from $V_1(x) < V(x)$ that $V_1(x)^4 < V(x)^4$ and therefore we could not assert that $H < H_1$.

2·3. – Let us consider now the sequence $\{V_n(x)\}$ of functions $V_n(x)$ determined in order by

$$V_{n+1}(x) = V_0(x) - \lambda^2 \int G(x - x_0) V_n(x_0)^3 \, \mathrm{d}x_0 , \tag{36}$$

with $n = 0, 1, 2, \ldots$ and $G(x - x_0)$ defined as in Eq. (25). The functions $V_n(x)$ exist under quite general conditions if $f(x) \to 0$ u.d. as $|x| \to \infty$ and $V_{n+1}(x)$ is the solution of the boundary problem

$$\Delta V_{n+1} - V_{n+1} + f - \lambda^2 V_n^3 = 0 , \qquad V_{n+1} \to 0 \quad \text{u.d. as} \quad |x| \to \infty . \tag{37}$$

From *Theor.* 2, Coroll. 1. [see § 1·2], we can in general only infer that, for $n = 0, 1, 2, \ldots$,

$$V_{2n} > V_{2n+1} < V_{2n+2} . \tag{38}$$

But suppose a positive number k can be found such that either $V_{2k-1}(x) < V_{2k+1}(x)$ or $V_{2k-2}(x) > V_{2k}(x)$ for *all* values of x. Then, from the value k on the sequence $\{V_{2n+1}(x)\}$ will increase monotonically and the sequence $\{V_{2n}(x)\}$ will be decreasing monotonic. Further, from that value on, any element of the former will be less than any element of the latter, and so both sequences are bounded and, being monotonic, both have a finite limit for all values of x. The first statement follows from Eq. (37) and the above mentioned corollary. The second statement follows from the first and the general relation (38).

Many questions arise now:

a) will the number k *always* exist?;

b) if k can be found, will be $\lim_{n\to\infty} V_{2n+1}(x) = \lim_{n\to\infty} V_{2n}(x)$ for all values of x?;

c) if both sequences have the same limit, say, $V(x)$, will $V(x)$ satisfy the integral equation

$$V(x) = V_0(x) - \lambda^2 \int G(x - x_0) V(x_0)^3 \, \mathrm{d}x_0 ? \tag{39}$$

The answer to the first question is no. The number k does not *always* exist. For suppose $f(x) \geqslant 0$, so that $Q(x)$ [see Appendix II] is everywhere positive and suppose moreover that Eq. (26) holds. Let λ be such that

$$\lambda^2 > \sup_x Q(x) = M.$$

We shall then have $V_1(x) \leqslant 0$ everywhere, and so, for all values of x,

$$V_2(x) - V_0(x) = -\lambda^2 \int G(x - x_0) V_1(x_0)^3 > 0 ,$$

which entails $V_1(x) > V_3(x)$, which entails $V_2(x) < V_4(x)$, and so on. Hence if $f(x) \geqslant 0$ and λ is large enough, so that (40) holds, the sequence $\{V_{2n+1}(x)\}$ will be decreasing monotonic and $\{V_{2n}(x)\}$ will increase monotonically from the value $n = 0$ on and for all values of x (fig. 2*a*).

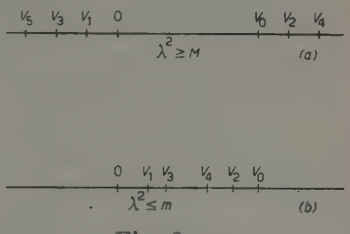

Fig. 2.

But suppose now that, under the same conditions holding for $f(x)$, we have $\lambda^2 \leqslant m$. Then $V_1(x) \geqslant 0$ and hence $V_2(x) < V_0(x)$ throughout the space. The second alternative above holds with $k = 1$, and so $\{V_{2n+1}(x)\}$ increases monotonically, $\{V_{2n}(x)\}$ is decreasing monotonic, and

$$0 < \lim_{n\to\infty} V_{2n+1}(x) = V(x) \leqslant \lim_{n\to\infty} V_{2n}(x) = V(x) < V_0(x) \tag{41}$$

for all values of x (fig. 2*b*). In this case, therefore, k exists.

We see therefore that, though k does not always exist, there is at least a case in which it does exist, namely when $f(x)$ ($\geqslant 0$) and λ satisfy Eqs. (26) and (32), respectively. But if these conditions actually hold, we can say much more. The reason will become apparent in the course of the subsequent section.

2·4. – Let $f(x)$ and λ satisfy the above mentioned conditions, so that for any positive integers m and n

$$0 < V_{2m+1}(x) < V(x) \leqslant V(x) < V_{2n}(x) < V_0(x) \quad \text{everywhere.} \tag{42}$$

Suppose further that

$$K \equiv 3\lambda^2 \max_x \int G(x - x_0) V_0(x_0)^2 \, dx_0 < 1 . \tag{43}$$

This can always be achieved by suitably chosing λ, since $\max_x \int G(x - x_0) \cdot V_0(x_0)^2 \, dx_0$ is finite and depends only on the structure of the source distribution, *but not* on λ. We have

$$\begin{aligned} V_{2n}(x) - V_{2n+1}(x) = \lambda^2 \int G(x - x_0)[V_{2n}(x_0)^2 + V_{2n}(x_0) V_{2n-1}(x_0) + V_{2n-1}(x_0)^2] + \\ + [V_{2n}(x_0) - V_{2n-1}(x_0)] \, dx_0 < \\ < 3\lambda^2 \int G(x - x_0) V_0(x_0)^2 [V_{2n-2}(x_0) - V_{2n-1}(x_0)] \, dx_0 , \end{aligned}$$

and so

$$\max_x (V_{2n}(x) - V_{2n+1}(x)) < K \max_x (V_{2n-2}(x) - V_{2n+1}(x)) , \tag{44}$$

where the quantities on both sides exist and are finite for all values of n. From (44) it follows that

$$\max_x (V_{2n} - V_{2n+1}) < K^n \max_x (V_0 - V_1) ,$$

where the expression on the right tends to zero as $n \to \infty$ $(K < 1)$.

Now

$$0 \leqslant \overline{V}(x) - \underline{V}(x) < V_{2n}(x) - V_{2n+1}(x) \leqslant \max_x (V_{2n} - V_{2n+1}) < K^n \max_x (V_0 - V_1) .$$

Hence, for all values of x,

$$\overline{V}(x) = \underline{V}(x) = V(x) , \tag{45}$$

and so both limits $\overline{V}$ and $\underline{V}$ coincide. Further, since

$$0 \leqslant V_{2n}(x) - V(x) \leqslant V_{2n}(x) - V_{2n+1}(x) < K^n \max_x (V_0 - V_1) ,$$

where the last expression on the right tends to zero as $n \to \infty$ and does not depend on x, the sequence $V_{2n}(x)$ tends to the limit $V(x)$ *uniformly* with regard to x throughout the space, and so

$$\lim_{n\to\infty} \int G(x - x_0) V_{2n}(x_0)^3 \, dx_0 = \int G(x - x_0) \{\lim_{n\to\infty} V_{2n}(x_0)^3\} dx_0 = \tag{46}$$

$$= \int G(x - x_0) V(x_0)^3 \, dx_0 \quad (*).$$

But, by Eqs. (36) and (45)

$$V(x) = \lim_{n\to\infty} V_{2n+1}(x) = V_0(x) - \lambda^2 \lim_{n\to\infty} \int G(x - x_0) V_{2n}(x_0)^3 \, dx_0 . \tag{47}$$

Hence, by Eqs. (46) and (47),

$$V(x) = V_0(x) - \lambda^2 \int G(x - x_0) V(x_0)^3 \, dx_0$$

and $V(x)$ is *the* solution of problem (P). We thus have the following

Theorem 3. If the source distribution $f(x)$ is non-negative, if $f(x) =$

(*) Note that $\int G(x - x_0)(V_{2n}^2 + V_{2n} V + V^2) dx < 3 \max V_0(x)^2$, and so the uniformity of the convergence allows us to assert that $\lim_{n\to\infty} \int^x = \int \lim_{n\to\infty}$.

$= 0(\exp[-r]/r^{\beta})$ with $\beta > 2$ as $r = |x| \to \infty$, uniformly with regard to the direction, and if

$$\lambda^2 < \inf\left(\inf_x \frac{V_0(x)}{\int G(x-x_0)V_0(x_0)^3\,\mathrm{d}x_0}\,,\ 3\max_x \int G(x-x_0)V_0(x_0)^2\,\mathrm{d}x_0\right),$$

where $V_0(x) = \int G(x-x_0)f(x_0)\,\mathrm{d}x_0$, the solution of problem (P) exists and is given by the common limit of the sequences $\{V_{2n+1}(x)\}$ and $\{V_{2n}(x)\}$, with the functions $V_n(x)$ determined in order by Eq. (36).

Consider now the integral

$$H_n = -\tfrac{1}{2}g^2\int(fV_n + \tfrac{1}{2}\lambda^2 V_n^4)\,\mathrm{d}x\,, \tag{48}$$

with n even or odd. This integral exists for all values of n, since $0 < V_n(x) < V_0(x)$ and the integral H_0 Eq. (34) is convergent. Now from Eqs. (42) and (45) it follows that

$$H_{2n} < H < H_{2n+1}\,,$$

where H_{2n} increases monotonically and H_{2n+1} decreases monotonically as $n \to \infty$. On the other hand

$$0 < H_{2n+1} - H_{2n} = \tfrac{1}{2}g^2\int[f + \tfrac{1}{2}\lambda^2(V_{2n}^2 + V_{2n+1}^2)(V_{2n} + V_{2n+1})](V_{2n} - V_{2n+1})\,\mathrm{d}x <$$

$$< \left\{\tfrac{1}{2}g^2\int(f + 2\lambda^2 V_0^3)\,\mathrm{d}x\right\}\cdot\max_x(V_{2n} - V_{2n+1})\,.$$

Hence, the integral $\frac{1}{2}g^2\int(f + 2\lambda^2 V_0^3)\,\mathrm{d}x$ being convergent,

$$\lim_{n\to\infty} H_{2n} = H = \lim_{n\to\infty} H_{2n+1}$$

and the error $H - H_{2n}$ (or $H_{2n+1} - H$) cannot be greater than a well defined quantity, namely

$$0 < H - H_{2n} < \left\{\tfrac{1}{2}K^n g^2\int(f + 2\lambda^2 V_0^3)\,\mathrm{d}x\right\}\cdot\max_x(V_0 - V_1)\,.$$

As in all cases when one has to do with *sufficient* conditions, the requirements to be met are perhaps too stringent from the physical point of view, particularly those to be met by λ. In fact, the series of iterations Eq. (36) may converge, for a particular source distribution $f(x)$, even when some of the sufficient conditions are not satisfied. In other words, our conclusions are too « pessimistic ». But we know by now that, at *least* for sufficiently small values of λ, the iteration procedure will lead to the solution of our problem.

2·5. – We shall in the following assume that $f(x)$ and λ are such that the considerations of the preceeding section apply. It follows from the definition Eq. (36) that $V_n(x)$ is a polinomial in λ^2 of degree $\nu(n) = (3^n - 1)/2$,

$$V_n(x) \equiv \sum_{m=0}^{\nu(n)} \lambda^{2m} V_n^{(m)}(x) . \tag{49}$$

The coefficients $V_n^{(m)}(x)$, which could be called *partial potentials* of the n-tieth approximation, have several properties that are readily proved by induction from Eqs. (36) and (38) and the properties assumed for the source distribution $f(x)$ [see *Theor.* 3]. Without going into the details of the proofs, we shall merely state these properties.

a) $V_n^{(0)}(x) \equiv V_0(x)$ for all values $n \geqslant 0$ and

$$V_n^{(1)}(x) \equiv V^{(1)}(x) \equiv -\int G(x - x_0) V_0(x_0)^3 \, \mathrm{d}x_0 \text{ for all } n \geqslant 1.$$

b) $V_n^{(m)}(x) \equiv 0$ for $m > \nu(n)$, by definition Eq. (49).

c) Given the partial potentials of the n-tieth approximation, those of the $(n+1)$-tieth approximation are given by

$$V_{n+1}^{(m)}(x) = -\int \mathrm{d}x_0 \, G(x - x_0) \sum_{m_1 + m_2 + m_3 = m-1} V_n^{(m_1)}(x_0) V_n^{(m_2)}(x_0) V_n^{(m_3)}(x_0) , \quad (m \geqslant 1) \tag{50}$$

where (m_1, m_2, m_3) is *any* partition of $m - 1$ into three sumands, all of them non-negative integers, and the $\sum$ is extended to all these partitions.

d) $V_n^{(m)}(x)$ is either identically zero or has the same sign as $(-1)^m$ for all values of n.

e) $V_n^{(m)}(x)$ does not depend on n for $n \geqslant m$ and is equal to $V_m^{(m)}(x)$. For simplicity we shall write $V^{(m)}(x)$ for this common value.

f) $|V_n^{(m)}(x)| < |V_{n+1}^{(m)}(x)|$ for all values $n < m$. In other words, in going from the n-tieth approximation to the approximation of order $n+1$, the absolute values of the partial potentials increase if $m > n$, and remain constant if $m \leqslant n$. In particular we find in the $(n+1)$-tieth approximation non-zero partial potentials which were identically zero in the preceding approximation, namely those for which $\nu(n) < m \leqslant \nu(n+1)$.

g) In any approximation and if $V_n^{(m)}(x)$ is not identically zero, then

$$V_n^{(m)}(x) \sim \frac{e^{-R}}{4\pi R} F_n^{(m)}(\boldsymbol{n}) \quad \text{u.d. as} \quad |x| \to \infty \quad \left(R \equiv |x| , \boldsymbol{n} \equiv \frac{\boldsymbol{R}}{R} \right) , \tag{51}$$

with $F_n^{(m)}(n)$ a continuous function on the surface of the sphere $|\boldsymbol{n}| = 1$. This result follows immediately from the *Lemma* in § 2·1, the asymptotic behavior of $f(x)$, and Eqs. (49) and (50).

Summarizing, the n-tieth approximation $V_n(x)$ is of the form

$$V_n(x) = V_0(x) + \lambda^2 V^{(1)}(x) \dots + \lambda^{2n} V^{(n)}(x) + \lambda^{2(n+1)} V_n^{(n+1)}(x) + \\ + \dots + \lambda^{2\nu(n)} V_n^{(\nu(n))} . \tag{52}$$

The partial potentials are alternately positive and negative, the first $V_0(x)$ being positive, and all behave as described by Eq. (51) as $|x| \to \infty$. The $n+1$ first partial potentials are common to the n-tieth approximation and all the subsequent approximations. It can easily be seen that this partial potentials are the $n+1$ first coefficients of a *formal* expansion of $V(x, \lambda)$ in powers of λ^2. The first partial potentials are as follows:

$$\left\{ \begin{aligned}
&V_0(x) = \int G(x - x_0) f(x_0)\, \mathrm{d}x_0 > 0 . \\
&V^{(1)}(x) = -\int G(x - x_0) V_0(x_0)^3\, \mathrm{d}x_0 < 0 , \\
&V^{(2)}(x) = -3 \int G(x - x_0) V_0(x_0)^2 V^{(1)}(x_0)\, \mathrm{d}x_0 > 0 , \\
&V_2^{(3)}(x) = -3 \int G(x - x_0) V_0(x_0) V^{(1)}(x_0)^2\, \mathrm{d}x_0 < 0 , \\
&V_2^{(4)}(x) = -\int G(x - x_0) V^{(1)}(x_0)^3\, \mathrm{d}x_0 > 0 , \\
&V^{(3)}(x) = -3 \int G(x - x_0) \{ V_0(x_0)^2 V^{(2)}(x_0) + V_0(x_0) V^{(1)}(x_0)^2 \} < 0 , \\
&V_3^{(4)}(x) = -\int G(x - x_0) \{ 3 V_0(x_0)^2 V_2^{(3)}(x_0) + 6 V_0(x_0) V^{(1)}(x_0) \cdot \\
&\qquad \cdot V^{(2)}(x_0) + V^{(1)}(x_0)^3 \} \mathrm{d}x_0 > 0 .
\end{aligned} \right. \tag{53}$$

The number of terms in each approximation increases very rapidly with n and the complexity of each term increases considerably with m. The present approach is thus not very promising from the practical point of view.

2·6. – Let us now introduce the expression Eq. (49) for $V_n(x)$ into the integral Eq. (48). We get for the energy in the n-tieth approximation

$$(54)\qquad H_n = -\frac{1}{2}g^2\int f(x)\,G(x-x_0)\,f(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 + \\ + \frac{1}{2}g^2\sum_{m=1}^{4\nu(n)+1}\lambda^{2m}\int \mathrm{d}x\Big\{V_0(x)\sum_{m_1+m_2+m_3=m-1} V_{n-1}^{(m_1)}(x)\,V_{n-1}^{(m_2)}(x)\,V_n^{(m_3)}(x) - \\ - \frac{1}{2}\sum_{m_1+m_2+m_3+m_4=m-1} V_n^{(m_1)}(x)\,V_n^{(m_2)}(x)\,V_n^{(m_3)}(x)\,V_n^{(m_4)}(x)\Big\}.$$

Note that the first sum will be zero for all values of m such that $\nu(n) < m \leqslant 4\nu(n)+1$ and that, as we said before, $V_n^{(m_i)} = 0$ $(V_{n-1}^{(m_i)} = 0)$ if $m_i > \nu(n)$ $(m_i > \nu(n-1))$. The explicit forms of the two first approximations are

$$(55)\qquad H_0(x) = -\frac{1}{2}g^2\int f(x)\,G(x-x_0)\,f(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 - \frac{1}{4}g^2\lambda^2\int V_0(x)^4\,\mathrm{d}x\,,$$

$$(56)\qquad H_1(x) = -\frac{1}{2}g^2\int f(x)\,G(x-x_0)\,f(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 + \frac{1}{4}g^2\lambda^2\int V_0(x)^4\,\mathrm{d}x - \\ - g^2\lambda^4\int V_0(x)^3 V^{(1)}(x)\mathrm{d}x - \frac{3}{2}g^2\lambda^6\int V_0(x)^2 V^{(1)}(x)^2\,\mathrm{d}x - \\ - g^2\lambda^8\int V_0(x)V^{(1)}(x)^3\,\mathrm{d}x - \frac{1}{4}g^2\lambda^{10}\int V^{(1)}(x)^4\,\mathrm{d}x\,.$$

Observe that in going from H_0 to H_1 the coefficient of λ^2 changes sign. As we shall presently see, the term in λ^2 in H_0 gives rise to an attraction, and so the term in λ^2 in the approximation H_1 shall determine a repulsion. As a matter of fact, the coefficient of λ^{2m} for given m will vary with n until $m \leqslant \nu(n-1)+1$. From this critical value of n on, the coefficient of λ^{2m} does not depend on n. For example, the coefficient of λ^2 will be the same as in Eq. (56) for all subsequent values of n, and the coefficient of λ^4 is equal to

$$\frac{1}{2}g^2\int V_0(x)^3 V^{(1)}\,\mathrm{d}x$$

from $n = 2$ on.

2·7. – Suppose now that the source distribution $f(x)$ is of the form Eq. (3), namely

$$f(x) = \sum_{i=1}^{A}\varrho(x-x_i)\,,$$

with the function $\varrho(x-x_i)$ as defined in § 1·1 and $\varrho(x) = 0\big(\exp[-r]/r^\beta\big)$ at

least ($\beta > 2$), so that also $f(x) = 0(\exp[-r]/r^{\beta})$. We shall have

$$(57) \qquad \int f(x)\,G(x-x_0)f(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 = \sum_{i=1}^{A}\int \varrho(x-x_i)\,G(x-x_0)\,\varrho(x_0-x_i)\,\mathrm{d}x\,\mathrm{d}x_0 + \\ + 2\sum_{i<j}^{A}\int \varrho(x-x_i)\,G(x-x_0)\,\varrho(x_0-x_j)\,\mathrm{d}x\,\mathrm{d}x_0\,,$$

$$(58) \qquad \int V_0(x)^4\,\mathrm{d}x = \sum_{i=1}^{A}\int V_0(x-x_i)^4\,\mathrm{d}x + \\ + 2\sum_{i<j}^{A}\int\{2V_0(x-x_i)^3V_0(x-x_j) + \\ + 2V_0(x-x_i)V_0(x-x_j)^3 + 3V_0(x-x_i)^2V_0(x-x_j)^2\}\,\mathrm{d}x + \\ + 12\sum_{i<j<k}^{A}\int\{V_0(x-x_i)^2V_0(x-x_j)V_0(x-x_k) - V_0(x-x_i)V_0(x-x_j)^2V_0(x-x_k) + \\ + V_0(x-x_i)V_0(x-x_j)V_0(x-x_k)^2\}\,\mathrm{d}x + \\ + 24\sum_{i<j<k<l}^{A}\int\{V_0(x-x_i)V_0(x-x_j)V_0(x-x_k)V_0(x-x_l)\}\mathrm{d}x\,,$$

where

$$V_0(x-x_i) \equiv \int G(x-x_0)\,\varrho(x_0-x_i)\,\mathrm{d}x_0\,.$$

The first term in Eqs. (57) and (58) are self-energy terms and are the only ones which do not tend to zero when the nucleons are an infinite distance apart each other. But the potential energy of the system of nucleons in each approximation is given by the difference $H_n - H_n(\infty)$, where $H_n(\infty)$ refers to the limit of H_n when *all* the distances between nucleons tend to infinity. The self-energy terms will therefore drop out, since they do not depend on the mutual distances. The second term in Eq. (57) represents the ordinary Yukawa two-body forces. A second kind of two-body forces proportional to $g^2\lambda^2$ is given by the second term Eq. (58). To be more precise, one would say that these forces are of two types as given by the sum of the first two terms and the third term of the integral, respectively. There is only one type of 3-body forces in this approximation and the same holds tor the 4-body forces. It was assumed, of course, that $A \geqslant 4$.

We see therefore that in the approximation H_0 we have to deal with 2, 3 and 4-body forces, all of them attractive. The same forces contribute to the H_1 approximation, but are then repulsive as pointed out before. New types of many body forces of increasing degree of complexity appear in the H_1 approximation. Even for the simplest case, that of the term in λ^4,

one finds three types of 2-body forces, three types of 3-body forces, two types of 4-body forces, one type of 5-body forces and one type of 6-body forces. The results are too involved to be quoted here. All these forces are attractive in the H_1 approximation and repulsive in the H_2 approximation. However, in the latter case the absolute value is $\frac{1}{2}$ of the absolute value in the H_1 approximation. Absolute value and the repulsive character are conserved in all subsequent approximations. That is, the effect of both terms, that proportional to λ^2 and that proportional to λ^4, is to diminish the attraction of the ordinary Yukawa two-body forces from the third approximation on.

The author is greatly indebted to Professor W. HEISENBERG, who suggested this problelem and offered helpful advice during its study.

APPENDIX I

Proof of the Lemma on page 145.

Take for simplicity the origin of the coordinate system at the point O. By a suitable choice of the constant R_0, we can secure, by Eq. (26), that for all values of x such that $|x| \geqslant R_0$ we shall have $|f(x)| < K(e^{-r}/r^{\beta})$, where K is a positive constant independent of the direction of the vector x. Take $R \geqslant 2R_0$ and write

$$\text{(A)} \qquad Re^R \int G(x - x_0) f(x_0)\, \mathrm{d}x_0 = Re^R \int_0^{R_0} r^2\, \mathrm{d}r \int_{\Omega} \frac{e^{-\varrho}}{4\pi\varrho} f(x_0)\, \mathrm{d}\Omega +$$

$$+ Re^R \int_{R_0}^{R} r^2 \mathrm{d}r \int_{\Omega} \frac{e^{-\varrho}}{4\pi\varrho} f(x_0)\, \mathrm{d}\Omega + Re^R \int_{R}^{\infty} r^2\, \mathrm{d}r \int_{\Omega} \frac{e^{-\varrho}}{4\pi\varrho} f(x_0)\, \mathrm{d}\Omega,$$

where $R \equiv |x|$ and $\varrho \equiv |x - x_0|$. Call for brevity I_1, I_2 and I_3 the first, second and third term on the right, respectively. It can be shown that in I_1

$$\frac{\exp[R - \varrho]}{4\pi(\varrho/R)} \to \frac{\exp[r \cos\psi]}{4\pi}, \qquad (r \equiv |x_0|)$$

as $R \to \infty$, uniformly with regard to x_0 throughout the sphere E_0 of center at the origin and radius R_0. ψ is the angle between the direction $\boldsymbol{n}$ along which x tends to infinity and that of the vector x_0. Now, by hypothesis,

$|f(x_0)|$ is integrable in E_0. Hence $\lim_{R\to\infty} I_1$ exists and is equal to

$$\frac{1}{4\pi}\int_0^{E_0} r^2\,\mathrm{d}r \int_{\Omega} \exp[r\cos\psi] f(x_0)\,\mathrm{d}\Omega\,.$$

Consider now I_2. It is easily proved from Eq. (26) that $F(R_0) \equiv \lim_{R\to\infty} I_2$ exists and is such that $|F(R_0)| < (K/(\beta-2))R_0^{2-\beta}$. Finally, $|I_3| < (K/2(\beta-2))R^{2-\beta}$ and, since $\beta > 2$, $I_3 \to 0$ as $R \to \infty$. We can thus assert that I_1, I_2 and I_3 tend to a finite limit as $R \to \infty$. Hence the expression on the left, Eq. (A), has a limit, too:

$$\text{(B)}\qquad \lim_{R\to\infty}\left\{R\,e^R \int G(x-x_0) f(x_0)\,\mathrm{d}x_0\right\} = \frac{1}{4\pi}\int_{E_0} \exp[r\cos\psi] f(x_0)\,\mathrm{d}x_0 + F(R_0)\,.$$

Now, the quantity on the left, Eq. (B), does not depend on R_0. We can therefore take the limit on the right and we get

$$\lim_{R\to\infty}\left\{R\,e^R \int G(x-x_0) f(x_0)\,\mathrm{d}x_0\right\} = \frac{1}{4\pi}\int \exp[r\cos\psi] f(x_0)\,\mathrm{d}x_0\,,$$

since $\lim_{R\to\infty} F(R_0) = 0$, and where the integral on the right is extended to all the space. This completes the proof of the *Lemma*.

Appendix II

If $f(x) \geqslant 0$ is at least $0(e^{-r}/r^\beta)$, with $\beta > 2$, uniformly with regard to the direction as $r \equiv |x| \to \infty$, then

$$m \equiv \inf_x \frac{V_0(x)}{\int G(x-x_0) V_0(x_0)^3\,\mathrm{d}x_0} > 0\,,$$

where $V_0(x)$ and $G(x-x_0)$ are defined as in Eq. (25).

To see this observe first that

$$Q(x) \equiv \frac{V_0(x)}{\int G(x-x_0) V_0(x_0)^3\,\mathrm{d}x_0}\,,$$

is a bounded continuous function of x throughout any finite domain and that, if $f(x) \geqslant 0$, $V_0(x)$ cannot be zero, so that $Q(x) > 0$ for all finite values of x. Now, by the *Lemma* above, $f(x) = 0(e^{-R}/R^\beta)$, with $R \equiv |x|$ and $\beta > 2$, entails $V_0(x)^3 = 0(e^{-3R}/R^3)$, so that $V_0(x)^3 = O(e^{-R}/R^3)$. But then, by the same

lemma (see Note on p. 145),

$$\lim_{R\to\infty} Q(x) \equiv \lim_{R\to\infty} \frac{V_0(x)}{\int G(x-x_0)V_0(x_0)^3\,\mathrm{d}x_0} = \frac{\int \exp[r\cos\psi]f(x_0)\,\mathrm{d}x_0}{\int \exp[r\cos\psi]V_0(x_0)^3\,\mathrm{d}x_0},$$

where the expression on the right is a function of the direction $\boldsymbol{n}$ along which x tends to infinity, everywhere *positive and continuous* on the surface $|\boldsymbol{n}| = 1$. Further, $Q(x)$ tends to its limit u.d.. Hence two positive numbers K and R_0 exist such that $Q(x) > K > 0$ for all $|x| > R_0$. Now, since $Q(x)$ is continuous and *positive* throughout the sphere $|x| \leqslant R_0$, we shall have $\min_{|x|\leq R_0} Q(x) > 0$ and therefore

$$m \equiv \inf_x Q(x) \geqslant \inf\left(K,\ \min_{|x|\leq R_0} Q(x)\right) > 0.$$

RIASSUNTO (*)

Nel primo paragrafo si derivano alcune proprietà generali delle soluzioni statiche dell'equazione di Schiff dalla struttura dell'equazione non lineare e dal comportamento della distribuzione delle sorgenti. Nel secondo paragrafo si dà la prova dell'esistenza della soluzione e si derivano le forze di più corpi implicite nella teoria scalare classica dei potenziali di Schiff.

(*) *Traduzione a cura della Redazione.*

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Multiple Pion Production Through Isomeric States of the Nucleon. Theorems Following from Charge Independence.

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Summary. — Some interpretations of the scattering and of the multiple production of pions based on the hypothesis of the intervention of isomeric states of the nucleons have recently been proposed. There are indications that this situation occurs, at least approximately, in certain energy ranges. Such a simplified interpretation of the complex phenomenon of multiple production may be insufficient, but on account of the importance of the problem it may nevertheless be useful to examine the main consequences. In this paper we consider multiple production in meson-nucleon, gamma-nucleon and nucleon-nucleon collisions. If the hypothesis is made that each time only the matrix elements corresponding to a given value of the total isotopic spin predominate, general relations can be obtained for the average numbers of pions of each sign produced which are independent of the multiplicity.

1. – Introduction.

There have been recently interpretations of the experiments on the scattering of pions by nucleons, on pion photo-production, and on pion production in nucleon-nucleon collisions based on the hypothesis of the existence of excited states of the nucleon (1). A precise definition of an excited nucleon was given

(1) K. A. Brueckner: *Phys. Rev.*, **86**, 106 (1952); K. A. Brueckner and K. M. Watson: *Phys. Rev.*, **86**, 923 (1952); G. S. James and W. L. Kraushaar: *Phys. Rev.*, **90**, 341 (1953); B. T. Feld: *Phys. Rev.*, **90**, 342 (1953); G. Wentzel: *Phys. Rev.*, **86**, 437 (1952).

by FERRETTI at the Bristol Conference in 1951 ([2]). FERRETTI moreover discussed the general implications of this definition. It is presumable that an adequate treatment of the pion-nucleon interaction will actually lead to the existence of these excited states. An adequate treatment of mesonic phenomena does not exist at present. However, it is known that the strong-coupling theory developed by PAULI and DANCOFF ([3]) gave indications of excited states of the nucleon. It is expected that an excited nucleon will decay after a very short mean life, into an ordinary nucleon with the emission of pions and γ-rays. It may occur that the mean life of the excited nucleon is sufficiently long to allow him, after being formed, to escape from the mesonic cloud, which is formed along with it, and then to decay with emission of pions. Such an excited state could be properly designated an « isomeric state of the nucleon ». Following a nomenclature proposed by BELINFANTE ([4]), we shall adopt the term *baryon* to indicate an « isomeric state of the nucleon ». The cross-section for the process in which a baryon is formed, and successively disintegrates, would be the product of the cross-section for the formation of the intermediate state of relatively long mean life by the probability that the intermediate state subsequently disintegrates into the final state considered.

In order to fix these ideas, let us first confider the collision between two nucleons. The production of pions can be considered to occur in part directly and in part through the decay of the baryons formed in the collision. The hypothesis that the production through baryon decay predominates over the direct production has been considered independently by various authours. BELINFANTE ([4]) has applied this hypothesis to nucleon-nucleon collisions in an energy interval from about 0.6 to 4.2 GeV. KRAUSHAAR and MARKS ([5]) have developed a theory of multiple production based on the hypothesis that the collision between two nucleons gives rise to two excited nucleons which successively decay as free particles. The model of KRAUSHAAR and MARKS contains quite detailed assumptions about the kinematics of the process of formation of the excited states. These authors assume that the excited nucleons will maintain the same direction of the incident nucleons and further they assume that the masses of the two excited nucleons are about equal. In any case it is remarkable how some of the general aspects of the phenomenon of multiple production, which seem to be more or less verified from the few experiments carried out, can be explained in a natural way by this model. Recently PEASLEE ([6]) has analysed solely on the basis of charge independence,

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([2]) See J. G. WILSON: *Progress in Cosmic Ray Physics*, II (Amsterdam, 1954), p. 104.
([3]) W. PAULI and S. M. DANCOFF: *Phys. Rev.*, **62**, 85 (1952).
([4]) F. J. BELINFANTE: *Phys. Rev.*, **92**, 145 (1953).
([5]) W. L. KRAUSHAAR and L. J. MARKS: *Phys. Rev.*, **93**, 326 (1954).
([6]) D. C. PEASLEE: *Phys. Rev.*, **94**, 1085 (1954).

some experiments on multiple production made by YUAN and LINDEBAUM (7) concluding in favour of an interpretation in terms of an excited state of the nucleon having isotopic spin 3/2. Some recent evidence in favour of the model comes from the results of the experiment of FOWLER, SHUTT, THORNDIKE and WHITTMORE (8). The evidence is based on the existence of angular correlations and is moreover consistent with the preference observed for double pion production over triple production.

The mean life of the baryon, although sufficiently long to permit it to cross a diameter of the order of the Compton wavelength of the meson, must nevertheless be sufficiently short so that the tracks of the pions produced and of the nucleons after the collision all seem, to come from the same point, as is observed in multiple production phenomena found in plates and in Wilson chambers. It is well known that Λ-particles, which also decay into a nucleon and a pion, have a much longer mean life. The problem of determining whether a Λ-particle may also be considered as a particular isomeric state of the nucleon, with a much longer mean life as a consequence of some still unknown circumstance, probably will constitute an important problem in the future for the theory of fields.

The case of production in meson-nucleon and γ-nucleon collisions is different in some respect from the case of nucleon-nucleon collisions. In this last case the excess kinetic energy after the baryon formation is taken by the two baryons, which, in the center of mass system, proceed with opposite momenta. In a meson-nucleon collision or in a γ-nucleon collision the real formation of a single baryon may occur just at the exact kinetic energy which compensates the mass difference between the baryon and the nucleon, for momentum conservation prohibits the process at a higher energy. Therefore one rather has to deal with a virtual resonance which extends in a certain energy interval. We picture this situation approximately by saying that, in that energy interval, the relevant matrix elements are those which lead to the formation of the resonant state. For what concerns the application of charge independence the two treatments in this case are equivalent. Of course, in the primary collision process a real baryon could well be formed together with other particles, as, for instance, pions, or possible excited states of the pion which, analogously to the baryons, successively decay into pions. We do not consider here processes of this kind.

In the following we shall apply the hypothesis of production through excited states of the nucleons to meson-nucleon collisions, γ-nucleon collisions, and nucleon-nucleon collisions. The hypothesis of charge independence leads

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(7) L. C. L. YUAN and S. L. LINDEBAUM: *Phys. Rev.*, **93**, 1431 (1954).

(8) W. B. FOWLER, R. P. SHUTT, A. M. THORNDIKE and W. L. WHITTMORE: Preprint.

to some general relations concerning the relative numbers of neutral and charged pions produced. Such relations are general in the sense that they do not depend on the order of multiplicity; further they do not depend on other specific assumptions about the production mechanism, except that of charge independence, and that of the production through isomeric states of the nucleons. For the case of direct production general relations have been given by WATSON (9) and by VAN HOVE, MARSHAK and PAIS (10). The model of production through excited states of the nucleons gives, of course, more specific relations. As a common feature these relations always involve the differences between the number of charged particles and twice the number of neutral particles produced in some particular states.

We would like to express at this point that the whole reasoning in any case is based on quite simple and general arguments, whose understanding presupposes only a general acquaintance with the concept of isotopic spin for nucleon-meson reactions. Throughout the entire discussion we have purposely avoided any reference to formalisms.

2. – Notation.

We indicate with b_T^M a baryon with isotopic spin T, and charge $(M+\frac{1}{2})e$. We consider the decay of a baryon b_T from a well defined state $\mathbf{i}$ (that is, with assigned momentum and polarization) into a nucleon in a state $\mathbf{b}$, into a meson in a state $\mathbf{c}^{(1)}$, into a meson in a state $\mathbf{c}^{(2)}$, ..., and into a meson in a state $\mathbf{c}^{(N)}$. In order to define completely the process, we must also specify the charge of the baryon, of the nucleon, of the meson which goes in $\mathbf{c}^{(1)}$, of the meson which goes in $\mathbf{c}^{(2)}$, ..., and of the meson which goes in $\mathbf{c}^{(N)}$. For example in order to designate the probability of disintegration of $b_{\frac{3}{2}}^+$ in state $\mathbf{i}$ into a proton in state $\mathbf{b}$, into a positive meson in state $\mathbf{c}^{(1)}$, and into a negative meson in state $\mathbf{c}^{(2)}$, we write

$$P(b_{\frac{3}{2}}^+ \,|\, p\,+\,-),$$

where the indication of the states $\mathbf{i}$, $\mathbf{b}$, $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$ is omitted. But, of course, the cross-section still depends on these states. We intend these states to be fixed once and for all when we compare processes which differ only in the charges of the particles that take part in the reaction. For example, $P(b_{\frac{3}{2}}^- \,|\, p\,-\,-)$ is the probability of disintegration of $b_{\frac{3}{2}}^-$ in state $\mathbf{i}$ into a proton in state $\mathbf{b}$, into a negative meson in state $\mathbf{c}^{(1)}$, and into a negative meson in state $\mathbf{c}^{(2)}$.

(9) K. M. WATSON: *Phys. Rev.*, **85**, 852 (1952).
(10) L. VAN HOVE, R. MARSHAK and A. PAIS: *Phys. Rev.*, **88**, 1211 (1952).

We now consider all the possible decay processes of the baryon b_T^M of charge $(M+\frac{1}{2})e$ in **i**, into a nucleon in **b**, and into N mesons in $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, ..., $\mathbf{c}^{(N)}$, and we sum all the decay probabilities relative to the processes in which the meson in $\mathbf{c}^{(1)}$ is positive, plus all the decay probabilities relative to the processes in which the meson in $\mathbf{c}^{(2)}$ is positive, ..., plus all the decay probabilities relative to the processes in which the meson in $\mathbf{c}^{(N)}$ is positive. We indicate this sum with

$$\nu(b_T^M|+)\,.$$

For a statistical ensemble of decay processes of b_T^M in state **i** into a nucleon in state **b**, the above expression is proportional to the mean number of positive mesons created in the states $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, ..., $\mathbf{c}^{(N)}$. In the following we will speak of such a mean number as « the mean number of positive mesons produced ».

Analogously we define $\nu(b_T^M|0)$ and $\nu(b_T^M|-)$ and analogous considerations hold. We consider, for example, the possible disintegrations of $b_{\frac{3}{2}}^+$ into a nucleon plus two mesons. The following disintegrations are possible

i	**b**	$\mathbf{c}^{(1)}$	$\mathbf{c}^{(2)}$
$b_{\frac{3}{2}}^+ \rightarrow$	p	$+$	$-$
	p	0	0
	p	$-$	$+$
	n	$+$	0
	n	0	$+$

For this case we will therefore have:

$$\nu(b_{\frac{3}{2}}^+|+) = P(b_{\frac{3}{2}}^+|p\,+\,-) + P(b_{\frac{3}{2}}^+|n\,+\,0) + P(b_{\frac{3}{2}}^+|p\,-\,+) + P(b_{\frac{3}{2}}^+|n\,0\,+)$$

$$\nu(b_{\frac{3}{2}}^+|\,0\,) = P(b_{\frac{3}{2}}^+|p\;0\;\;0\,) + P(b_{\frac{3}{2}}^+|n\,0\;\;+) + P(b_{\frac{3}{2}}^+|p\;0\;\;0\,) + P(b_{\frac{3}{2}}^+|n\,+\,0)$$

$$\nu(b_{\frac{3}{2}}^+|-) = P(b_{\frac{3}{2}}^+|p\,-\,+) + P(b_{\frac{3}{2}}^+|p\,+\,-)\,.$$

We denote with $\bar{\nu}(b_T^M|+)$ (and analogously for the neutral and negative pions) the sum of $\nu(b_T^M|+)$ carried out over all the possible final states **b** of the nucleon, and over all the possible groups of N final states $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, ..., $\mathbf{c}^{(N)}$ of the mesons (each group counted only once). It is clear that $\bar{\nu}(b_T^M|+)$ is proportional to the number of positive mesons produced when the baryon b_T^M in state **i** decays into a nucleon and into N pions.

In order to illustrate how such a procedure functions, we refer to the case above of the decay of the $b_{\frac{3}{2}}^+$ into a nucleon with the emission of two mesons. We take for example the second of the equations listed above. Consider the sum of $\nu(b_{\frac{3}{2}}^+|0)$ over all the possible final states **b** of the nucleon and over all the possible pairs of final states $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$ for the two mesons (each pair counted

only once). The sum on $[P(b^+_{\frac{3}{2}} | n\,0\,+) + P(b^+_{\frac{3}{2}} | n\,+\,0)]$ gives the total probability of the decay of $b^+_{\frac{3}{2}}$ in state **i** into a neutron and into one neutral and one charged meson. In fact, in the sum are included all the pairs of final states $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$ for the two mesons, but each pair is counted twice, once with $P(b^+_{\frac{3}{2}} | n\,0+)$ and once with $P(b^+_{\frac{3}{2}} | n\,+\,0)$, and it is as if we had summed independently over all the possible $\mathbf{c}^{(1)}$ and all the possible $\mathbf{c}^{(2)}$. This is exactly the correct procedure in this case since the two mesons are of different charge, and thus distinguishable. Carrying out instead, the sum on $(Pb^+_{\frac{3}{2}} | p\,0\,0)$, one obtains the total probability of the decay of $b^+_{\frac{3}{2}}$ in state **i**, into a proton and two neutral mesons, since in this case the two mesons are indistinguishable. Therefore the final result is given by:

$$\bar{\nu}(b^+_{\frac{3}{2}} | 0) = 2P(b^+_{\frac{3}{2}} | p\,0\,0)_{\text{total}} + P(b^+_{\frac{3}{2}} | n\,0\,+)_{\text{total}}$$

and as one sees $\bar{\nu}(b^+_{\frac{3}{2}} | 0)$ is proportional to the mean number of neutral mesons produced. We have given a detailed clarification on these points so that it will be clear in the following that *all the relations which we obtain are valid not only for the mean number of mesons of a given sign produced in certain states (that is, with certain momenta and certain polarizations) but also for the total mean numbers of mesons of a given sign produced.* This follows from the fact that such relations are linear.

We now introduce the difference:

$$\Delta(b^M_T) = \nu(b^M_T | +) - 2\nu(b^M_T | 0) + \nu(b^M_T | -)$$

which is proportional to the difference between the (mean) number of charged pions produced and twice the (mean) number of neutral pions produced.

3. – The Decay of the Baryon.

We consider the decay of the baryon b_T in state **i**, into a nucleon in state **b**, and into N mesons in the states $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, ..., $\mathbf{c}^{(N)}$.

The relation:

$$\sum_{M=-T}^{T} \nu(b^M_T | +) = \sum_{M=-T}^{T} \nu(b^M_T | 0) \tag{1}$$

follows immediately from charge independence.

The simplest way to recognize the validity of (1) is the following. One considers a statistical ensemble of decay processes in which a baryon b_T in state **i** decays into a nucleon in state **b**, with the emission of N mesons in states $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, ..., $\mathbf{c}^{(N)}$. One supposes that in this statistical ensemble all pos-

sible charge states for b_T are represented with equal probability. In such a case (1) simply expresses the property of the charge independent interaction of not introducing privileged directions in charge space.

It is convenient to substitute for (1) the equivalent form:

$$\sum_{M=\frac{1}{2}}^{T} \Delta(b_T^M) = 0 \tag{2}$$

which follows from the relations of charge symmetry:

$$v(b_T^M \mid m) = v(b_T^{-M} \mid -m) \qquad (m = +, 0, -)$$

From (3) it further follows that

$$\Delta(b_T^M) = \Delta(b_T^{-M}) . \tag{4}$$

4. – Some Particular Cases.

In the case of an isomeric state of the nucleon with $T=\frac{1}{2}$ for which there are the two charge states $b_{\frac{1}{2}}^+$ and $b_{\frac{1}{2}}^0$, Eq. (2) and (4) give the following relations (of which the second is the charge symmetric of the first:

$$\Delta(b_{\frac{1}{2}}^+) = 0 , \qquad \Delta(b_{\frac{1}{2}}^0) = 0 . \tag{5}$$

For the decay with the emission of a single meson, the first equation of (5) is written:

$$P(b_{\frac{1}{2}}^+ \mid n+) - 2P(b_{\frac{1}{2}}^+ \mid p\,0) = 0$$

which can be verified by a detailed study of the process. For decay with emission of two mesons the same equation of (5) is written:

$$2[P(b_{\frac{1}{2}}^+ \mid p\,+\,-) + P(b_{\frac{1}{2}}^+ \mid p\,-\,-\,+)] = P(b_{\frac{1}{2}}^+ \mid n\,0\,+) + P(b_{\frac{1}{2}}^+ \mid n\,+\,0) + 4P(b_{\frac{1}{2}}^+ \mid p\,0\,0)$$

which can be similarly verified. In the case of an isomeric state of the nucleon with $T=\frac{3}{2}$, for which there are the four possible charge states $b_{\frac{3}{2}}^{++}$, $b_{\frac{3}{2}}^+$, $b_{\frac{3}{2}}^0$ and $b_{\frac{3}{2}}^-$, Eqs. (2) and (4) give the following relations, of which the last two follow from charge symmetry:

$$\begin{cases} \Delta(b_{\frac{3}{2}}^{++}) + \Delta(b_{\frac{3}{2}}^+) = 0 \\ \Delta(b_{\frac{3}{2}}^0) = \Delta(b_{\frac{3}{2}}^+) \\ \Delta(b_{\frac{3}{2}}^-) = \Delta(b_{\frac{3}{2}}^{++}) . \end{cases} \tag{6}$$

For decay with emission of a single meson, the first of (6) is written

$$P(b_{\frac{3}{2}}^{+}|n+) - 2P(b_{\frac{3}{2}}^{+}|p\,0) = -P(b_{\frac{3}{2}}^{++}|p+)$$

which can be verified. For the decay with emission of two mesons, the first of (6) is written:

$$2[P(b_{\frac{3}{2}}^{+}|p+-) + P(b_{\frac{3}{2}}^{+}|p-+)] - [P(b_{\frac{3}{2}}^{+}|n0+) + P(b_{\frac{3}{2}}^{+}|n+0)] - 4P(b_{\frac{3}{2}}^{+}|p00) -$$
$$- [P(b_{\frac{3}{2}}^{++}|p+0) + P(b_{\frac{3}{2}}^{++}|p\,0+)] + 2P(b_{\frac{3}{2}}^{++}|n++) = 0$$

which can be verified by a detailed study of the process.

5. – Meson-Nucleon Collisions.

We wish here to discuss multiple production through baryon formation in pion-nucleon collisions.

The initial state formed by a pion and a nucleon contains at most the two states of total isotopic spin 1/2 and 3/2. We can thus have excited isomeric states of the nucleon $b_{\frac{1}{2}}$ or $b_{\frac{3}{2}}$.

We introduce a notation completely analogous to that used to describe the disintegration of the baryon. That is, we refer to processes in which the initial particles are found in a well defined state **a**, the final nucleon in a well defined state **b**, and the N final mesons in the well defined states $\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(N)}$. With $\sigma(+n|p+--)$, for example, we denote the cross-section of the process in which one finds initially a positive meson and a nucleon in state **a**, and at the end one has a nucleon in state **b**, a positive meson in state $\mathbf{c}^{(1)}$ and a negative meson in state $\mathbf{c}^{(2)}$.

Further $\nu(+n|+)$, for example, denotes the sum of all the cross-sections relative to the processes in which the meson in $\mathbf{c}^{(1)}$ is positive, plus all the cross-sections relative to the processes in which the meson in $\mathbf{c}^{(2)}$ is positive, ... plus all cross-sections in which the meson in $\mathbf{c}^{(N)}$ is positive.

For a statistical ensemble of collisions between a positive meson and a neutron in state **a**, which give rise to a nucleon in state **b**, the preceding sum is proportional to the mean number of positive mesons produced in the states $\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(N)}$. In the following, we will speak of such a mean number as the mean number of mesons produced of a given sign. The remarks made for the case of the decay of the baryon about the total probabilities of disintegration are valid also in this case for the total cross-sections.

We introduce again the difference

$$\Delta(A) = \nu(A|+) - 2\nu(A|0) + \nu(A|-)\,,$$

where A denotes $+p$, $0p$, $-p$, $+n$, $0n$, $-n$ according to the case.

Further we introduce the notation.

$$\nu[p\,|\,+] = \nu(+p\,|\,+) + \nu(0\,p\,|\,+) + \nu(-p\,|\,+)$$

and the analogous notations $\nu[p\,|\,0]$, $\nu[p\,|-]$, $\nu[n\,|\,+]$, $\nu[n\,|\,0]$, $\nu[n\,|-]$. For example, $\nu[p\,|\,+]$ is proportional to the number of positive mesons produced in a statistical ensemble in which the initial meson has an equal probability of being positive, negative, or neutral.

Moreover we define

$$\Delta[p] = \nu[p\,|\,+] - 2\nu[p\,|\,0] + \nu[p\,|-]$$

and analogously $\Delta[n]$.

The sum of all the cross-sections, always relative to the states $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, ..., $\mathbf{c}^{(x)}$, which have A as the initial state, we call $\sigma(A)$.

We first discuss the case of multiple production through the formation of $b_{\frac{1}{2}}$ and its successive decay.

The relations

$$\sigma(+p) = \sigma(-n) = 0\,, \qquad 2\sigma(0\,p) = \sigma(-p) = \sigma(+n) = 2\sigma(0\,n) \tag{7}$$

follow from the hypothesis that production occurs through the formation of $b_{\frac{1}{2}}$. From (5) one gets immediately

$$\Delta(A) = 0, \tag{8}$$

where A can signify $0\,p$, $-p$, $0\,n$, and $+n$. *Therefore, if production occurs through an isomeric state of the nucleon with isotopic spin* 1/2, *then, for any order of multiplicity, the number of charged mesons produced is twice the number of neutral mesons produced whatever the charges of the colliding meson and of the nucleon.* Due to the linearity of (8), this conclusions is valid also when one considers simultaneously various processes of different multiplicity which are possible at a given energy. From (8), adding the relations obtained for the various A's, one gets

$$\Delta[p] = 0\,, \qquad \Delta[n] = 0\,. \tag{9}$$

Relations (9) are valid also without the hypothesis that the production occurs through excited states of the nucleon. This has been demonstrated by Van Hove, Marshak and Pais ([9]).

We now discuss the case of multiple production through the formation of $b_{\frac{3}{2}}$ and its successive decay.

For this case, the relations

$$2\sigma(+p) = 3\sigma(0p) = 6\sigma(-p) = 6\sigma(+n) = 3\sigma(0n) = 2\sigma(-n) \tag{10}$$

follow from the hypothesis that production occurs through the formation of $b_{\frac{3}{2}}$ and its successive decay. From the preceding relations (10) and from (6) one gets the relations

$$\left\{\begin{array}{l} \Delta(0p) = -\frac{2}{3}\Delta(+p) \\ \Delta(-p) = -\frac{1}{3}\Delta(+p) \\ \Delta(+n) = -\frac{1}{3}\Delta(+p) \\ \Delta(0n) = -\frac{2}{3}\Delta(+p) \\ \Delta(-n) = \Delta(+p)\,. \end{array}\right. \tag{11}$$

We demonstrate, for example, the first of (11), the demonstration of the remaining relations being completely analogous. We calculate first, for example, $\nu(0p\,|\,+)$. This is proportional to the number of positive mesons produced (in the sense explained above). Now, the number of positive mesons produced will be proportional to the number of collisions that produce $b_{\frac{3}{2}}^{+}$ in state **i**, multiplied by the number of positive mesons produced when $b_{\frac{3}{2}}^{+}$ decays from state **i**, the whole summed over all states **i**. The number of such collisions is proportional in our case to the cross-section, $\sigma(0p\,|\,b_{\frac{3}{2}}^{+})$, while the number of positive mesons produced when $b_{\frac{3}{2}}^{+}$ decays from state **i** is proportional to $\nu(b_{\frac{3}{2}}^{+}\,|\,+)$. Combining the two constants of proportionality (which depend only on the statistical ensemble being considered) in a simple constant $\mathcal{C}$, we obtain

$$\nu(0p\,|\,+) = \mathcal{C}\sum_{\mathbf{i}}\sigma(0p\,|\,b_{\frac{3}{2}}^{+})\nu(b_{\frac{3}{2}}^{+}\,|\,+) = \tfrac{2}{3}\mathcal{C}\sum_{\mathbf{i}}\sigma(+p\,|\,b_{\frac{3}{2}}^{++})\nu(b_{\frac{3}{2}}^{+}\,|\,+)$$

from which

$$\Delta(0p) = \tfrac{2}{3}\mathcal{C}\sum_{\mathbf{i}}\sigma(+p\,|\,b_{\frac{3}{2}}^{++})\Delta(b_{\frac{3}{2}}^{+}) = -\tfrac{2}{3}\mathcal{C}\sum_{\mathbf{i}}\sigma(+p\,|\,b_{\frac{3}{2}}^{++})\Delta(b_{\frac{3}{2}}^{++})$$

and considering the analogous expression for $\Delta(+p)$ one gets the first equation of (11), as we had wished. Therefore, *if prodution occurs through an isomeric state of the nucleon with isotopic spin* 3/2, *then the difference between the number of charged mesons produced and twice the number of neutral mesons produced, once known for one of the six possible charge states of the meson and of the initial nucleon,* $+p$, $0p$, $-p$, $+n$, $0n$, $-n$, *is also known for the remaining cases, for any order of multiplicity, from relations* (11). Due to the linearity of Eqs. (11) this conclusion is also valid considering simultaneously the various processes of different multiplicity possible at the given energy. From (10) one gets the

relations

$$\Delta[p] = 0\,, \qquad \Delta[n] = 0$$

which, as has been noted, have a more general character, in the sense that they do not depend on the hypothesis of production through isomeric states of the nucleon.

If the hypothesis that production occurs through isomeric states of the nucleon is valid, at least in certain energy intervals, the relations (7), (8), (10) and (11) must hold and ought to be very useful given their generality and simplicity. Eqs. (7) and (10) not only can serve to verify the hypothesis of production through excited states of the nucleon, but also to determine the isotopic spin of such excited states. The most simple and expressive interpretation of the relations derived above is, clearly, that in terms of the « total average numbers of mesons produced ». The more detailed interpretation refers to the cross-sections for particles in well defined states. For completeness, we give an example in the appendix of the detailed use of the above relations, considering the simple case of single meson production in meson-nucleon collisions (meson-nucleon scattering would be the most simple example, of course, but it is perhaps too simple to be illustrative here).

6. – Multiple Photoproduction.

Charge independence is not valid for the first part of the process, production of the baryon; it is valid, however for the second part of the process, decay of the baryon.

In the case of multiple photoproduction through the formation of $b_{\frac{1}{2}}$ and its successive decay, one has from (5)

$$\Delta(\gamma p) = 0\,, \qquad \Delta(\gamma n) = 0 \tag{12}$$

that is, *the number of charged mesons produced equals twice the number of neutral mesons at any order of multiplicity.*

In his work on charge independence ([9]), WATSON shows how the interaction Hamiltonian for this problem can be thought of as the sum of one part T_0, which behaves as a scalar for rotations in the isotopic spin space, plus a part T_1^0, which behaves as the zero component of a spherical vector. Therefore, the only isomeric states of the nucleon that can be excited should have isotopic spin 1/2 or 3/2. In the case of production of $b_{\frac{1}{2}}$, T_0 as well as T_1^0 have matrix elements different from zero; on the other hand, for production of $b_{\frac{3}{2}}$ only the matrix element of T_1^0 enters. Because of this, the cross-sections for production of $b_{\frac{1}{2}}^+$ and $b_{\frac{1}{2}}^0$ will be in general different, but the cross-sections for production of $b_{\frac{3}{2}}^+$ and of $b_{\frac{3}{2}}^0$ are equal. We can therefore

conclude that, *for multiple photoproduction through the formation of* $b_{\frac{3}{2}}$ *and its successive decay a relation of the type*

$$\Delta(\gamma p) = \Delta(\gamma n) \tag{13}$$

will be valid. This relation follows from eqs. (4) derived from charge-symmetry and from the model assumed for the production of the baryon, using the same arguments as for the meson-nucleon case.

7. – Multiple Production in Nucleon-Nucleon Collisions.

For completeness, we extend the preceding ideas to the case of nucleon-nucleon collisions, although, as we shall see, the results which will be obtained are already contained in the general relation of Watson.

The initial state formed by the two nucleons is, in general, a fixed superposition of the two states with isotopic spin 1 and 0. In the case that only one of the two nucleons is excited, the isotopic spin of this excited state can be only $\frac{1}{2}$ or $\frac{3}{2}$. However, in the case in which both the nucleons are excited, the isotopic spins of these excited states can have any values however large provided that they differ from each other by at most one unit. We shall consider a particular example. Let us consider the case in which both nucleons are excited in states $b_{\frac{3}{2}}$, which successively decay, giving rise to multiple production of mesons. As before we will designate, for example, with $\sigma(pp|b^{++}b^0)$ the cross-section for the process in which the two initial nucleons are found in a certain total state **a** and, at the end, a baryon of charge 2 is found in a certain state **i**, and a baryon of charge 0 is found in a certain state **i'**. We indicate, as before, for example, with $\Delta(b^{++})$ the expression $\nu(b^{++}|+) - 2\nu(b^{++}|0) + \nu(b^{++}|-)$, where it is understood that b^{++} decays from the state **i** into a nucleon and into a certain number of mesons that are found in a total state **C**, and we denote with $\Delta'(b^{++})$ the same expression with the only difference that this time b^{++} decays instead from **i'**. As before, we put $\Delta(pp) = \nu(pp,+) - 2\nu(pp|0) + (pp|-)$ and analogously for $\Delta(pn)$. Following the same reasoning as for the case of meson-nucleon collisions, we can write, omitting the usual constant of proportionality that depends on the statistical ensemble being considered,

$$\Delta(pp) = \sum_{(\mathbf{i},\mathbf{i}')} \{\sigma(pp|b^{++}b^0)[\Delta(b^{++}) + \Delta'(b^0)] + \sigma(pp|b^0b^{++})[\Delta(b^0) + \Delta'(b^{++})] + \sigma(pp|b^+b^+)[\Delta(b^+) + \Delta'(b^+)]\}$$

$$\Delta(pn) = \sum_{(\mathbf{i},\mathbf{i}')} \{\sigma(pn|b^{++}b^-)[\Delta(b^{++}) + \Delta'(b^-)] + \sigma(pn|b^-b^{++})[\Delta(b^-) + \Delta'(b^{++})] + \sigma(pn|b^+b^0)[\Delta(b^+) + \Delta'(b^0)] + \sigma(pn|b^0b^+)[\Delta(b^0) + \Delta'(b^+)]\} .$$

In the sum expression the indices are placed in parentheses to indicate that the sum should be taken over all the possible pairs of intermediate states **i**, **i'**, each pair being counted only once. By writing in this way the possible indistinguishability of the two baryons produced is already accounted for in the correct way. One notes that, under the first sum $\sigma(pp\,|\,b^+b^0)$ and $\sigma(pp\,|\,b^0b^{++})$ are counted as distinct, while $\sigma(pp\,|\,b^+b^+)$ is counted only once.

Taking into account the relationships (6), and expressing explicity the various σ's in terms of the two independent matrix elements for isotopic spin 1 and 0, one can easily verify the relation

$$\Delta(pp) + \Delta(pn) = 0\,.$$

This relation is nothing but the well known relation of Watson which is valid, as is known, also without the hypothesis of production through isobaric states.

I wish to express sincere thanks to Prof. B. Ferretti for the interest in this work and for the useful conversations.

Appendix

We consider the process

$$\text{pion} + \text{nucleon} \to \text{nucleon} + \text{pion} + \text{pion}\,.$$

Suppose first that the matrix elements relative to states having total isotopic spin $\frac{1}{2}$ predominate. In this case relations (8) hold, which may be written:

$$2[\sigma(0p\,|\,p\,+\,-) + (0p\,|\,p\,-\,+)] - 4\sigma(0p\,|\,p00) - [\sigma(0p\,|\,n\,+\,0) + \sigma(0p\,|\,n0\,+)] = 0$$

$$-[\sigma(-\,p\,|\,p0\,-) + \sigma(-\,p\,|\,p\,-\,0)] + 2[\sigma(-\,p\,|\,n\,+\,-) + \sigma(-\,p\,|\,n\,-\,+)] - \\ - 4\sigma(-\,p\,|\,n00) = 0$$

and the corresponding charge-symmetrical relations.

If the matrix elements relative to states having total isotopic spin $\frac{3}{2}$ predominate, relations (11) hold, which may be written:

$$2[\sigma(0p\,|\,p\,+\,-) + \sigma(0p\,|\,p\,-\,+)] - 4\sigma(0p\,|\,p00) - [\sigma(0p\,|\,n\,+\,0) + \sigma(0p\,|\,n0\,+)] - \\ - \tfrac{2}{3}[\sigma(+\,p\,|\,p\,+\,0) + \sigma(+\,p\,|\,p0\,+)] + \tfrac{4}{3}\sigma(+\,p\,|\,n\,+\,+) = 0$$

$$-[\sigma(-\,p\,|\,p0\,-) + \sigma(-\,p\,|\,p\,-\,0)] + 2[\sigma(-\,p\,|\,n\,+\,-) + \sigma(-\,p\,|\,n\,-\,+)] - \\ - 4\sigma(-\,p\,|\,n00) - \tfrac{1}{3}[\sigma(+\,p\,|\,p\,+\,0) + \sigma(+\,p\,|\,p0\,+)] + \tfrac{2}{3}\sigma(+\,p\,|\,n\,+\,+) = 0$$

and the corresponding charge symmetrical relations. Note that the cross sections relative to processes in which the space-spin states of the final mesons are interchanged appear always together. This just corresponds to the fact that the interference terms between the different matrix elements cancel each other in the above expressions.

The validity of the relations given above may be easily verified using the expressions for the cross-sections of these particular processes which may be obtained by a more detailed study. We report here for completeness these cross-sections, which refer to particles in well defined states.

$$\sigma(+\,p\,|\,p\,+\,0) = |\sqrt{\tfrac{1}{10}}\,\mathcal{S}_{\frac{3}{2}} - \sqrt{\tfrac{1}{2}}\,\mathcal{A}_{\frac{3}{2}}|^2$$

$$\sigma(0\;\;p\,|\,p\,+\,-) = |\sqrt{\tfrac{2}{45}}\,\mathcal{S}_{\frac{3}{2}} - \sqrt{\tfrac{2}{9}}\,\mathcal{A}_{\frac{3}{2}} - \sqrt{\tfrac{1}{18}}\,\mathcal{A}_{\frac{1}{2}} + \sqrt{\tfrac{1}{9}}\,\mathcal{S}_{\frac{1}{2}}|^2$$

$$\sigma(+\,p\,|\,p\,0\;\;+) = |\sqrt{\tfrac{1}{10}}\,\mathcal{S}_{\frac{3}{2}} + \sqrt{\tfrac{1}{2}}\,\mathcal{A}_{\frac{3}{2}}|^2$$

$$\sigma(0\;\;p\,|\,p\,0\;\;0) = |\sqrt{\tfrac{8}{45}}\,\mathcal{S}_{\frac{3}{2}}|^2$$

$$\sigma(-\,p\,|\,p\;\;0\,-) = |\sqrt{\tfrac{1}{10}}\,\mathcal{S}_{\frac{3}{2}} - \sqrt{\tfrac{1}{18}}\,\mathcal{A}_{\frac{3}{2}} - \sqrt{\tfrac{2}{9}}\,\mathcal{A}_{\frac{1}{2}}|^2$$

$$\sigma(0\;\;p\,|\,p\,-\,+) = |\sqrt{\tfrac{2}{45}}\,\mathcal{S}_{\frac{3}{2}} + \sqrt{\tfrac{2}{9}}\,\mathcal{A}_{\frac{3}{2}} + \sqrt{\tfrac{1}{18}}\,\mathcal{A}_{\frac{1}{2}} + \sqrt{\tfrac{1}{9}}\,\mathcal{S}_{\frac{1}{2}}|^2$$

$$\sigma(-\,p\,|\,p\,-\,0) = |\sqrt{\tfrac{1}{10}}\,\mathcal{S}_{\frac{3}{2}} + \sqrt{\tfrac{1}{18}}\,\mathcal{A}_{\frac{3}{2}} + \sqrt{\tfrac{2}{9}}\,\mathcal{A}_{\frac{1}{2}}|^2$$

$$\sigma(+\,p\,|\,n\,+\,+) = |\sqrt{\tfrac{4}{5}}\,\mathcal{S}_{\frac{3}{2}}|^2$$

$$\sigma(0\;\;p\,|\,n\,+\,0) = |-\sqrt{\tfrac{1}{5}}\,\mathcal{S}_{\frac{3}{2}} - \sqrt{\tfrac{1}{9}}\,\mathcal{A}_{\frac{3}{2}} + \sqrt{\tfrac{1}{9}}\,\mathcal{A}_{\frac{1}{2}}|^2$$

$$\sigma(-\,p\,|\,n\,+\,-) = |-\sqrt{\tfrac{1}{45}}\,\mathcal{S}_{\frac{3}{2}} - \sqrt{\tfrac{1}{9}}\,\mathcal{A}_{\frac{3}{2}} + \sqrt{\tfrac{1}{9}}\,\mathcal{A}_{\frac{1}{2}} + \sqrt{\tfrac{2}{9}}\,\mathcal{S}_{\frac{1}{2}}|^2$$

$$\sigma(0\;\;p\,|\,n\;\;0\;\;+) = |-\sqrt{\tfrac{1}{5}}\,\mathcal{S}_{\frac{3}{2}} + \sqrt{\tfrac{1}{9}}\,\mathcal{A}_{\frac{3}{2}} - \sqrt{\tfrac{1}{9}}\,\mathcal{A}_{\frac{1}{2}}|^2$$

$$\sigma(-\,p\,|\,n\;\;0\;\;0) = |-\sqrt{\tfrac{4}{45}}\,\mathcal{S}_{\frac{3}{2}} - \sqrt{\tfrac{2}{9}}\,\mathcal{S}_{\frac{1}{2}}|^2$$

$$\sigma(-\,p\,|\,n\,-\,+) = |-\sqrt{\tfrac{1}{45}}\,\mathcal{S}_{\frac{3}{2}} + \sqrt{\tfrac{1}{9}}\,\mathcal{A}_{\frac{3}{2}} - \sqrt{\tfrac{1}{9}}\,\mathcal{A}_{\frac{1}{2}} + \sqrt{\tfrac{2}{9}}\,\mathcal{S}_{\frac{1}{2}}|^2 .$$

A matrix element $\mathcal{S}_T$ refers to a final state symmetric in the two pions and having total isotopic spin T; a matrix element $\mathcal{A}_T$ to a final state antisymmetric having total spin T.

The matrix elements for the case of double photoproduction have been evaluated in another connection [11]. The expressions for $\Delta(\gamma p)$ and $\Delta(\gamma n)$ are as follows

$$\Delta(\gamma p) = 2[\sigma(\gamma p\,|\,p\,+\,-) + \sigma(\gamma p\,|\,p\,-\,+)] - 4\sigma(\gamma p\,|\,p00) - \\ - [\sigma(\gamma p\,|\,n\,+\,0) + \sigma(\gamma p\,|\,n0\,+)]$$

$$\Delta(\gamma n) = -[\sigma(\gamma n\,|\,p0\,-) + \sigma(\gamma n\,|\,p\,-\,0)] + \\ + 2[\sigma(\gamma n\,|\,n\,+\,-) + \sigma(\gamma n\,|\,n\,-\,+)] - 4\sigma(\gamma n\,|\,n00) .$$

The validity of equations (12) and (13) may easily be verified.

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[11] R. GATTO: *Nuovo Cimento*, **12**, 568 (1954).

RIASSUNTO

Sono state proposte recentemente delle interpretazioni dei fenomeni di diffusione di pioni e di produzione multipla basate sulla ipotesi dell'intervento di stati isomerici dei nucleoni. Si hanno attualmente indicazioni che una tale situazione si verifichi, almeno approssimativamente, in certi intervalli di energia. Può darsi che questa interpretazione piuttosto semplificata di un fenomeno complesso come la produzione multipla risulti molto incompleta. Tuttavia, data l'importanza del problema, può essere opportuno svolgerne le principali conseguenze. In questo lavoro vengono considerati i processi di produzione multipla in urti mesone-nucleone, gamma-nucleone e nucleone-nucleone. Se si fa l'ipotesi che essenzialmente intervengano volta a volta solo gli elementi di matrice relativi ad un ben determinato valore dello spin isotopico totale, seguono delle relazioni generali, cioè indipendenti dalla molteplicità, che legano tra di loro i numeri medii di pioni di vario segno che vengono prodotti.

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Transformation of a Frequency Equation in Corrugated Wave Guide Theory.

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Summary. — The frequency equation resulting from an approximate theory given by Walkinshaw, has been generalized and exactly transformed into a new formula still containing an infinite series. The same mathematical transformation can also be applied to the frequency equation after having introduced a suitable approximation to simplify the general term of the series on the right hand side. This leads to the desired approximate equation in a closed form. Since the mentioned approximation depends on the choice of four parameters, a discussion is presented on the possible ways to obtain a sufficiently high accuracy.

1. – Introduction.

In a recent note (1), Combe has presented some approximate formulae to compute the breadth of the lowest pass band in periodically irisloaded wave guides with circular symmetry. His calculations start with a frequency equation established by Walkinshaw (2), namely

$$\frac{F_1(ka)}{F_0(ka)} = \frac{d}{D} \sum_{m=-\infty}^{\infty} \frac{k}{\chi_m} \frac{J_1(\chi_m a)}{J_0(\chi_m a)} \frac{J_0(\beta_m d/2) \sin(\beta_m d/2)}{\beta_m d/2} \tag{1}$$

in which we keep the same notations as those used by the mentioned authors. Going over to the 0 and the π mode, Combe simply truncates the series in (1)

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(1) R. Combe: *Compt. Rend. Paris*, **238**, 1697 (Avril 1954).
(2) W. Walkinshaw: *Proc. Phys. Soc., London*, **61**, 246 (1948).

keeping respectively one and two terms in the formula. If such a truncation method were applied to the case of an arbitrary mode, one could generally not expect to obtain a sufficiently accurate summation of the series which seems to be slowly converging, most of the time. As an example, let us consider Table I containing thirteen terms of the series (not including the factor d/D) in the following special case corresponding to a $\pi/2$ mode:

$$D = 23.93 \text{ mm}; \qquad d = 21.43 \text{ mm}; \qquad a = 13.32 \text{ mm};$$

$$\lambda_0 = 10 \text{ cm}; \qquad \beta_0 = \frac{2\pi}{\lambda_g} = \frac{\pi}{2D}.$$

TABLE I.

m	Terms in eq. (1)	m	Terms in eq. (1)
0	0.335 965		
1	0.006 790	— 1	0.016 820
2	0.000 170	— 2	0.005 066
3	— 0.000 266	— 3	0.002 416
4	— 0.000 099	— 4	0.001 222
5	0.000 080	— 5	0.000 571
6	0.000 181	— 6	0.000 213

The sum of the terms is 0.36913, differing by roughly 10% from the term $m = 0$. Several terms have to be added together in order to obtain a much better approximation for the infinite series.

A detailed analysis of the general case leads to the conclusion, that the method consisting in keeping only one or two terms of the series in (1) gives rise to final results which are subjected to a variable inaccuracy.

Although eq. (1) results from an approximate theory in which the tangential electric field at the corrugation mouths has been conveniently chosen as

$$E'_z = \frac{C \exp[-i\beta_0 nD]}{[1-(2x_n/d)^2]^{\frac{1}{2}}} \qquad \left(-\frac{d}{2} \leqslant x_n \leqslant \frac{d}{2}\right), \tag{2}$$

it seems particularly interesting to study the possibility of summing its right hand side in a closed form which should be at once simple and very accurate. The presentation of such an approximation procedure constitutes the subject of this paper. Before starting the calculations we wish to emphasize that:

1) The summation will be carried out for any value of β_0, so as to include all possible modes in the pass band.

2) Equation (1) is only valid under the assumption (2), although it may be important to introduce other trial functions for E'_z in the theory. The

simplest way to make this possible consists in rewriting the theory starting with a tangential field E_z' which contains an arbitrary complex function.

According to a theorem about the symmetry properties of E_z in a periodic wave guide ([3]), we may generally write in the n-th corrugation:

$$(3) \qquad E_z' = \left[g_1\left(\frac{2x_n}{d}\right) + ig_2\left(\frac{2x_n}{d}\right)\right] \exp\left[-i\beta_0 nD\right] \qquad \left(-\frac{d}{2} \leqslant x_n \leqslant \frac{d}{2}\right)$$

in which $g_1(x)$ and $g_2(x)$ are two real functions existing in the interval $(-1 \leqslant x \leqslant 1)$, being respectively symmetrical and antisymmetrical with respect to $x = 0$ and possibly depending on certain factors appearing in the problem.

Using eq. (3), the theory gives rise to the following generalized frequency equation:

$$(4) \qquad G_1(0)\frac{F_1(ka)}{F_0(ka)} = \frac{d}{D}\sum_{m=-\infty}^{\infty}\frac{k}{\chi_m}\frac{J_1(\chi_m a)}{J_0(\chi_m a)}\left[G_1\left(\frac{\beta_m d}{2}\right) - G_2\left(\frac{\beta_m d}{2}\right)\right]\frac{\sin(\beta_m d/2)}{(\beta_m d/2)},$$

in which we have introduced

$$(5) \qquad G_1(x) \equiv \int_0^1 g_1(z)\cos xz\,\mathrm{d}z\,, \qquad G_2(x) \equiv \int_0^1 g_2(z)\sin xz\,\mathrm{d}z\,.$$

Our summation method will be applied to eq. (4) without specifying the functions $g_1(x)$ and $g_2(x)$.

In the next paragraphs, we shall first of all present an exact transformation of (4) into an entirely new formula still containing an infinite series on the right hand side. After this, it will be shown how the same transformation procedure can be applied together with an approximation.

2. – Exact Transformation of the Generalized Frequency Equation.

Let us consider the infinite product representing the Bessel function $J_0(z)$ in the entire complex z-plane ([4]):

$$(6) \qquad J_0(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\alpha_n^2}\right),$$

in which $\alpha_1, \alpha_2, \ldots$ represent the positive roots of $J_0(x) = 0$.

Taking the logarithm of both sides and differentiating with respect to z,

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([3]) W. Walkinshaw and J. S. Bell: *A.E.R.E. Report T/R 864* (1952).

([4]) G. N. Watson: *Theory of Bessel Functions* (Cambridge, 1948), p. 498.

we get after having divided by $-z$:

$$\frac{J_1(z)}{zJ_0(z)} = 2\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2 - z^2}. \tag{7}$$

Replacing z by $\chi_m a$, the left hand side becomes exactly the function which we wish to replace in the series of (4). Therefore, we can write:

$$G_1(0)\frac{F_1(ka)}{F_0(ka)} = \frac{2kad}{D}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(\alpha_n^2 - \chi_m^2 a^2)}\left[G_1\left(\frac{\beta_m d}{2}\right) - G_2\left(\frac{\beta_m d}{2}\right)\right]\frac{\sin(\beta_m d/2)}{\beta_m d/2} = \tag{8}$$

$$= \frac{2kad}{D}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(K_n^2 + \beta_m^2 a^2)}\left[G_1\left(\frac{\beta_m d}{2}\right) - G_2\left(\frac{\beta_m d}{2}\right)\right]\frac{\sin(\beta_m d/2)}{\beta_m d/2}$$

in which we have made use of the relation $\chi_m^2 = k^2 - \beta_m^2$. We have also introduced the quantities K_n defined by

$$K_n \equiv \sqrt{\alpha_n^2 - k^2 a^2}. \tag{9}$$

First, we shall carry out the exact summation of the series

$$S_1 \equiv \sum_{m=-\infty}^{\infty}\frac{1}{(K^2 + \beta_m^2 a^2)}\,G_1\left(\frac{\beta_m d}{2}\right)\frac{\sin(\beta_m d/2)}{\beta_m d/2}, \tag{10}$$

where K is an arbitrary constant.

Let us define

$$s_1(x) \equiv \sum_{m=-\infty}^{\infty}\frac{1}{(K^2 + \beta_m^2 a^2)}\,G_1\left(\frac{\beta_m d}{2}\right)\cos\frac{\beta_m d}{2}x \qquad (0 \leqslant x \leqslant 1), \tag{11}$$

so that

$$S_1 = \int_0^1 s_1(x)\,\mathrm{d}x. \tag{12}$$

Making use of the integral representation (5) for $G_1(x)$ and introducing

$$\beta_m = \beta_0 + \frac{2\pi m}{D},$$

we get

$$s_1(x) = \int_0^1 g_1(z)\sum_{m=-\infty}^{\infty}\frac{\cos(\beta_m d/2)z\cos(\beta_m d/2)x}{K^2 + \beta_m^2 a^2}\,\mathrm{d}z = \tag{13}$$

$$= \frac{1}{2}\int_0^1 g_1(z)\sum_{m=-\infty}^{\infty}\frac{\cos(\beta_m d/2)(z+x) + \cos(\beta_m d/2)(z-x)}{K^2 + \beta_m^2 a^2}\,\mathrm{d}z =$$

$$= \frac{D^2}{8\pi^2 a^2} \int_0^1 g_1(z) \sum_{m=-\infty}^{\infty} \frac{\cos(\pi d/D)[m + (\beta_0 D/2\pi)](z+x)}{(KD/2\pi a)^2 + [m + (\beta_0 D/2\pi)]^2} \, dz +$$

$$+ \frac{D^2}{8\pi^2 a^2} \int_0^1 g_1(z) \sum_{m=-\infty}^{\infty} \frac{\cos(\pi d/D)[m + (\beta_0 D/2\pi)](z-x)}{(KD/2\pi a)^2 + [m + (\beta_0 D/2\pi)]^2} \, dz .$$

In Appendix I, it is shown that

$$(14) \qquad \sum_{m=-\infty}^{\infty} \frac{\cos(m+B)x}{A^2 + (m+B)^2} = \frac{\pi}{A} \frac{\sinh A(2\pi - |x|) + \cos 2B\pi \sinh A|x|}{\cosh 2A\pi - \cos 2B\pi}$$

valid in the interval $(-2\pi \leqslant x \leqslant 2\pi)$. The formula must be applied replacing x respectively by

$$\frac{\pi d}{D}(z+x) \qquad \text{and by} \qquad \frac{\pi d}{D}(z-x) .$$

Knowing that

$$0 \leqslant z \leqslant 1 \qquad \text{and} \qquad 0 \leqslant x \leqslant 1 ,$$

we get

$$0 \leqslant \frac{\pi d}{D}(z+x) \leqslant 2\pi \frac{d}{D} \leqslant 2\pi$$

$$-\pi \leqslant -\frac{\pi d}{D} \leqslant \frac{\pi d}{D}(z-x) \leqslant \frac{\pi d}{D} \leqslant \pi$$

showing that (14) is applicable in the mentioned interval. Eq. (13) becomes:

$$(15) \quad s_1(x) = \frac{D}{4aK[\cosh(KD/a) - \cos\beta_0 D]} \int_0^1 g_1(z) \left\{ \sinh \frac{KD}{a} \left[1 - \frac{d}{2D}(z+x) \right] + \right.$$

$$\left. + \sinh \frac{KD}{a} \left[1 - \frac{d}{2D}|z-x| \right] + \cos\beta_0 D \left[\sinh \frac{Kd}{2a}(z+x) + \sinh \frac{Kd}{2a}|z-x| \right] \right\} dz =$$

$$= \frac{D}{2aK[\cosh(KD/a) - \cos\beta_0 D]} \left[\sinh \frac{KD}{a} \left(1 - \frac{d}{2D} x \right) \int_0^x g_1(z) \cosh \frac{Kd}{2a} z \, dz + \right.$$

$$+ \cosh \frac{Kd}{2a} x \int_x^1 g_1(z) \sinh \frac{KD}{a} \left(1 - \frac{d}{2D} z \right) dz +$$

$$+ \cos\beta_0 D \cdot \sinh \frac{Kd}{2a} x \int_0^x g_1(z) \cosh \frac{Kd}{2a} z \, dz +$$

$$\left. + \cos\beta_0 D \cdot \cosh \frac{Kd}{2a} x \int_x^1 g_1(z) \sinh \frac{Kd}{2a} z \, dz \right]$$

and we find

$$(16)\quad S_1 = \frac{D}{2aK[\cos\mathrm{h}(KD/a) - \cos\beta_0 D]}\Bigg[\int_0^1 \sinh\frac{KD}{a}\left(1-\frac{d}{2D}x\right)\mathrm{d}x\int_0^x g_1(z)\cosh\frac{Kd}{2a}z\,\mathrm{d}z +$$

$$+\int_0^1 \cosh\frac{Kd}{2a}x\,\mathrm{d}x\int_x^1 g_1(z)\sinh\frac{KD}{a}\left(1-\frac{d}{2D}z\right)\mathrm{d}z +$$

$$+\cos\beta_0 D\int_0^1 \sinh\frac{Kd}{2a}x\,\mathrm{d}x\int_0^x g_1(z)\cosh\frac{Ka}{2a}z\,\mathrm{d}z +$$

$$+\cos\beta_0 D\int_0^1 \cosh\frac{Kd}{2a}x\,\mathrm{d}x\int_x^1 g_1(z)\sinh\frac{Kd}{2a}z\,\mathrm{d}z\Bigg].$$

Each of these double integrals can be simplified by partial integration with respect to x:

$$(17)\quad \int_0^1 \sinh\frac{KD}{a}\left(1-\frac{d}{2D}x\right)\mathrm{d}x\int_0^x g_1(z)\cosh\frac{Kd}{2a}z\,\mathrm{d}z =$$

$$= -\frac{2a}{Kd}\int_0^1 \mathrm{d}\cosh\frac{KD}{a}\left(1-\frac{d}{2D}x\right)\int_0^x g_1(z)\cosh\frac{Kd}{2a}z\,\mathrm{d}z =$$

$$= -\frac{2a}{Kd}\cosh\frac{KD}{a}\left(1-\frac{d}{2D}\right)\int_0^1 g_1(z)\cosh\frac{Kd}{2a}z\,\mathrm{d}z +$$

$$+\frac{2a}{Kd}\int_0^1 g_1(x)\cosh\frac{Kd}{2a}x\cosh\frac{KD}{a}\left(1-\frac{d}{2D}x\right)\mathrm{d}x$$

and in the same way

$$(18)\quad \int_0^1 \cosh\frac{Kd}{2a}x\,\mathrm{d}x\int_x^1 g_1(z)\sinh\frac{KD}{a}\left(1-\frac{d}{2D}z\right)\mathrm{d}z =$$

$$= \frac{2a}{Kd}\int_0^1 g_1(x)\sinh\frac{KD}{a}\left(1-\frac{d}{2D}x\right)\sinh\frac{Kd}{2a}x\,\mathrm{d}x,$$

$$(19)\quad \int_0^1 \sinh\frac{Kd}{2a}x\,\mathrm{d}x\int_0^x g_1(z)\cosh\frac{Kd}{2a}z\,\mathrm{d}z =$$

$$= \frac{2a}{Kd}\cosh\frac{Kd}{2a}\int_0^1 g_1(z)\cosh\frac{Kd}{2a}z\,\mathrm{d}z - \frac{2a}{Kd}\int_0^1 g_1(x)\cosh^2\frac{Kd}{2a}x\,\mathrm{d}x,$$

$$\int_0^1 \cosh\frac{Kd}{2a}\,x\,\mathrm{d}x \int_x^1 g_1(z)\sinh\frac{Kd}{2a}\,z\,\mathrm{d}z = \frac{2a}{Kd}\int_0^1 g_1(x)\sinh^2\frac{Kd}{2a}\,x\,\mathrm{d}x\,. \tag{20}$$

Putting these results into (16), we obtain after some simplifications the final result for the series S_1:

$$S_1 = \frac{D}{dK^2}\left[G_1(0) - \frac{\cosh(KD/a)[1-(d/2D)] - \cos\beta_0 D\cosh(Kd/2a)}{\cosh(KD/a)-\cos\beta_0 D}\,G_1\left(\frac{iKd}{2a}\right)\right]. \tag{21}$$

The summation of the series

$$S_2 \equiv \sum_{m=-\infty}^{\infty}\frac{1}{(K^2+\beta_m^2 a^2)}\,G_2\left(\frac{\beta_m d}{2}\right)\frac{\sin(\beta_m d/2)}{\beta_m d/2} \tag{22}$$

can be carried out in a very similar way. Again we define

$$s_2(x) \equiv \sum_{m=-\infty}^{\infty}\frac{1}{(K^2+\beta_m^2 a^2)}\,G_2\left(\frac{\beta_m d}{2}\right)\cos\frac{\beta_m d}{2}\,x \qquad (0\leqslant x\leqslant 1) \tag{23}$$

so that

$$S_2 = \int_0^1 s_2(x)\,\mathrm{d}x\,, \tag{24}$$

and just as before, (23) can be transformed into

$$s_2(x) = \frac{D^2}{8\pi^2 a^2}\int_0^1 g_2(z)\sum_{m=-\infty}^{\infty}\frac{\sin(\pi d/D)[m+(\beta_0 D/2\pi)](z+x)}{(KD/2\pi a)^2+[m+(\beta_0 D/2\pi)]^2}\,\mathrm{d}z + \tag{25}$$

$$+\frac{D^2}{8\pi^2 a^2}\int_0^1 g_2(z)\sum_{m=-\infty}^{\infty}\frac{\sin(\pi d/D)[m+(\beta_0 D/2\pi)](z-x)}{(KD/2\pi a)^2+[m+(\beta_0 D/2\pi)]^2}\,\mathrm{d}z\,.$$

Using the result of Appendix II, namely

$$\sum_{m=-\infty}^{\infty}\frac{\sin(m+B)x}{A^2+(m+B)^2} = \frac{\pi}{A}\,\frac{\sin 2B\pi\sinh Ax}{\cosh 2A\pi-\cos 2B\pi} \qquad (-2\pi\leqslant x\leqslant 2\pi), \tag{26}$$

we get

$$s_2(x) = \frac{D\sin\beta_0 D}{4aK[\cosh(KD/a)-\cos\beta_0 D]}\cdot \tag{27}$$

$$\cdot\int_0^1 g_2(z)\left[\sinh\frac{Kd}{2a}(z+x)+\sinh\frac{Kd}{2a}(z-x)\right]\mathrm{d}z =$$

$$= \frac{D \sin \beta_0 D \cosh (Kd/2a)x}{2aK[\cosh (KD/a) - \cos \beta_0 D]} \int_0^1 g_2(z) \sinh \frac{Kd}{2a} z \, dz =$$

$$= \frac{D \sin \beta_0 D \cosh (Kd/2a)x \, G_2(iKd/2a)}{2iaK[\cosh (KD/a) - \cos \beta_0 D]} .$$

Applying (24), we find for the series S_2:

$$S_2 = \frac{D}{idK^2} - \frac{\sin \beta_0 D \sinh (Kd/2a)}{[\cosh (KD/a) - \cos \beta_0 D]} G_2\left(\frac{iKd}{2a}\right) . \tag{28}$$

The results (21) and (28) can be brought into (8) and we obtain directly

$$G_1(0)\frac{F_1(ka)}{F_0(ka)} = 2ka \sum_{n=1}^{\infty} \frac{1}{K_n^2} \left[G_1(0) - \right. \tag{29}$$

$$\left. - \frac{\left\{\cosh \frac{K_n D}{a}\left[1 - \left(\frac{d}{2D}\right)\right] - \cos \beta_0 D \cosh \frac{K_n d}{2a}\right\} G_1\left(\frac{iK_n d}{2a}\right) - i \sin \beta_0 D \sinh \frac{K_n d}{2a} G_2\left(\frac{iK_n d}{2a}\right)}{\cosh \frac{K_n D}{a} - \cos \beta_0 D} \right] .$$

Taking into account that

$$\sum_{n=1}^{\infty} \frac{1}{K_n^2} = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2 - k^2 a^2} = \frac{J_1(ka)}{2ka J_0(ka)} , \tag{30}$$

eq. (29) can be written as

$$\frac{1}{2ka} G_1(0)\left[\frac{F_1(ka)}{F_0(ka)} - \frac{J_1(ka)}{J_0(ka)}\right] = \tag{31}$$

$$= - \sum_{n=1}^{\infty} \frac{\cosh (K_n D/a)[1 - (d/2D)] - \cos \beta_0 D \cosh (K_n d/2a)}{K_n^2[\cosh (K_n D/a) - \cos \beta_0 D]} G_1\left(\frac{iK_n d}{2a}\right) +$$

$$+ i \sin \beta_0 D \sum_{n=1}^{\infty} \frac{\sinh (K_n d/2a)}{K_n^2[\cosh (K_n D/a) - \cos \beta_0 D]} G_2\left(\frac{iK_n d}{2a}\right) .$$

Using the definition of F_0 and F_1, the left hand side can be simplified:

$$\frac{F_1(ka)}{F_0(ka)} - \frac{J_1(ka)}{J_0(ka)} = \frac{J_1(ka)Y_0(kb) - Y_1(ka)J_0(kb)}{J_0(ka)Y_0(kb) - Y_0(ka)J_0(kb)} - \frac{J_1(ka)}{J_0(ka)} = \tag{32}$$

$$= \frac{[J_1(ka)Y_0(ka) - Y_1(ka)J_0(ka)]J_0(kb)}{[J_0(ka)Y_0(kb) - Y_0(ka)J_0(kb)]J_0(ka)} =$$

$$= \frac{2J_0(kb)}{\pi ka J_0(ka)[J_0(ka)Y_0(kb) - Y_0(ka)J_0(kb)]} ,$$

in which the wellknown relation

$$\begin{vmatrix} J_0(z) & Y_0(z) \\ \frac{\mathrm{d}J_0}{\mathrm{d}z} & \frac{\mathrm{d}Y_0}{\mathrm{d}z} \end{vmatrix} = \frac{2}{\pi z} \tag{33}$$

has been taken into account. Hence, the transformed frequency equation takes the following form:

$$\frac{J_0(kb)G_1(0)}{\pi(ka)^2 J_0(ka)F_0(ka)} = \sum_{n=1}^{\infty} \frac{\cosh(K_n D/a)[1-(d/2D)] - \cos\beta_0 D \cosh(K_n d/2a)}{K_n^2[\cosh(K_n D/a) - \cos\beta_0 D]} G_1\left(\frac{iK_n d}{2a}\right) - i\sin\beta_0 D \sum_{n=1}^{\infty} \frac{\sinh(K_n d/2a)}{K_n^2[\cosh(K_n D/a) - \cos\beta_0 D]} G_2\left(\frac{iK_n d}{2a}\right). \tag{34}$$

When E_z' is represented by (2) as in Walkinshaw's theory, we have

$$G_1(x) = C\int_0^1 \frac{\cos xz\,\mathrm{d}z}{\sqrt{1-z^2}} = \frac{\pi}{2} C J_0(x), \qquad G_2(x) = 0 \tag{35}$$

so that

$$G_1(0) = \frac{\pi}{2} C, \qquad G_1\left(\frac{iK_n d}{2a}\right) = \frac{\pi}{2} C I_0\left(\frac{K_n d}{2a}\right), \tag{35'}$$

and the frequency equation becomes

$$\frac{J_0(kb)}{\pi(ka)^2 J_0(ka)F_0(ka)} = \sum_{n=1}^{\infty} \frac{\cosh(K_n D/a)[1-(d/2D)] - \cos\beta_0 D \cosh(K_n d/2a)}{K_n^2[\cosh(K_n D/a) - \cos\beta_0 D]} I_0\left(\frac{K_n d}{2a}\right), \tag{36}$$

which is completely equivalent to (1). In both formulae (34) and (36), the phase angle $\beta_0 D$ characterizing the mode of oscillation is entirely separated from the other quantities appearing in the problem .

3. – Approximate Transformation of the Frequency Equation.

The exact transformation is based on the possibility of replacing $J_1(z)/zJ_0(z)$ by an infinite series which is valid in the entire complex plane. Instead of this, if we could introduce an approximation being the sum of a few terms of the same form as those appearing in (7) and representing $J_1(z)/zJ_0(z)$ sufficiently well in a relatively extended region, we would directly obtain a good approximation for the frequency equation. The most straigthforward way to try this would certainly consist in keeping only a few terms on the right hand

side of (7). However, to obtain a reasonable accuracy, one would still need too many terms. A suitable modification has to be introduced.

Let us consider the quantity

$$\chi_m = \sqrt{k^2 - \beta_m^2} \qquad (m = 0, \pm 1, \pm 2, \ldots).$$

In almost every practical application of wave guide theory, as e.g. in the design of linear electron accelerators, χ_m is a purely imaginary quantity for all values of m. This can easily be seen when one considers the orders of magnitude of the wavelengths λ_0, λ_g and the guide dimension D. In some other cases, when one studies wave propagation at phase velocities equal to or larger than c, χ_m becomes real for only one or two values of m, most of the time. This shows that the function $J_1(z)/zJ_0(z)$ must be particularly well approximated for imaginary values of z. When

$$\frac{I_1(x)}{xI_0(x)} \equiv \frac{J_1(ix)}{ixJ_0(ix)} \tag{37}$$

is plotted vs. x in the interval $(0 \leqslant x \leqslant 20)$, we find a bell-shaped curve which we can try to approximate by means of one or more functions of the type

$$\frac{A}{B^2 + x^2}. \tag{38}$$

The numerical calculations prove that (37) can be very well represented by a sum of two terms like (38) with suitably chosen coefficients:

$$\frac{I_1(x)}{xI_0(x)} \cong \frac{A_1}{B_1^2 + x^2} + \frac{A_2}{B_2^2 + x^2}. \tag{39}$$

In fact, the coefficients A_1, A_2, B_1 and B_2 can be calculated in such a way that the approximate curve has four chosen points in common with the exact one. This is shown in Appendix III which contains the equations enabling us to calculate the coefficients. One of the intersection points may eventually have a purely imaginary abscissa, when it is required that (39) should also be valid for imaginary values of x in the vicinity of the origin. The remaining question concerns the choice of the four common points. There are two interesting possibilities to discuss:

1) Among all the possible values of

$$\frac{J_1(\chi_m a)}{\chi_m a J_0(\chi_m a)} \qquad (m = 0, \pm 1, \ldots), \tag{40}$$

one can pick out the four largest ones corresponding to four values of $\chi_m a$.

Using the equations

$$\chi_m a = ix\,, \qquad \frac{J_1(\chi_m a)}{\chi_m a J_0(\chi_m a)} = y\,, \tag{41}$$

we find four sets of values (x, y) defining the four points in question. It is obvious that with this procedure, the approximate sum of the right hand side of (1) or (4) will include the exact values of four terms (probably the largest ones) of the series, together with approximate values for the remaining smaller terms. There is even more to say about the 0 and the π mode. Indeed, in the first case, we know that $\beta_0 = 0$ and therefore

$$\chi_0 = k\,, \qquad \chi_m = \chi_{-m} \qquad (m = 1, 2, \ldots),$$

showing that seven terms characterized by $m = 0, \pm 1, \pm 2, \pm 3$ will be exactly taken into account. In the case of the π mode, we have

$$\beta_m = -\beta_{-m-1} \quad \text{so that} \quad \chi_m = \chi_{-m-1}\,,$$

and we see that eight terms ($m = 0, \pm 1, \pm 2, \pm 3, -4$) will enter the calculations with their exact values.

The approximation method becomes more and more applicable as the intervals separating the abscissae of the four chosen points increase and this happens with increasing values of the ratio a/D. However, the procedure implies the recalculation of the constants A_1, A_2, B_1 and B_2 for each special case and the formulae of Appendix III show that this will involve a considerable amount of computation.

2) In practical applications, it may occur that one wishes to apply the frequency equation for a certain mode in a relatively large number of wave guides with dimensions which are not too different comparing one guide to another one in the set. Then it is generally possible to use the same suitably chosen constants A_1, A_2, B_1 and B_2 in all the cases. They can, for instance, be calculated applying the preceding method to the guide having dimensions equal to the average values of those appearing in the set and using an estimate of the average of the frequencies to be obtained.

As a typical example of form. (39), let us examine the case in which the curves coincide at $x = 0$, 1, 3 and 10. Application of Appendix III leads to

$$\frac{I_1(z)}{zI_0(z)} \cong \frac{2.2390948}{6.0409706 + z^2} + \frac{17.1548583}{132.6251039 + z^2}\,. \tag{42}$$

The constants appearing in this approximation have been successfully used in all the calculations which we have carried out to check the experimental

values of the guide diameters $2b$ for the first thirty cavities of our linear electron accelerator. In table II, we compare both sides of (42) for some real and some imaginary values of z. The agreement can be regarded as satisfactory, except in the vicinity of the first zero of $J_0(x)$. However, this is generally unimportant and in some exceptional cases, this difficulty can be eliminated, if necessary.

Replacing $J_1(z)/zJ_0(z)$ by

$$\frac{A_1}{B_1^2 - z^2} + \frac{A_2}{B_2^2 - z^2} \tag{43}$$

TABLE II.

x	$I_1(x)/xI_0(x)$	Approxim.	x	$J_1(x)/xJ_0(x)$	Approxim.
0	0.500 000 0	0.500 000 0	0	0.500 000 0	0.500 000 0
0.5	0.484 999 2	0.485 027 1	0.2	0.502 516 6	0.502 509 6
1	0.446 389 9	0.446 389 9	0.4	0.510 274 3	0.510 240 3
2	0.348 887 4	0.348 557 4	0.6	0.523 939 1	0.523 840 0
3	0.269 995 0	0.269 995 0	0.8	0.544 794 2	0.544 548 4
4	0.215 880 6	0.217 011 5	1.0	0.575 080 9	0.574 510 5
5	0.178 676 7	0.180 966 8	1.2	0.618 716 6	0.617 425 4
6	0.152 060 0	0.154 993 6	1.4	0.682 900 3	0.679 956 0
7	0.132 218 9	0.135 132 5	1.6	0.782 132 7	0.775 133 1
8	0.116 904 4	0.119 214 9	1.8	0.950 229 2	0.931 987 2
9	0.104 743 4	0.106 028 2	2.0	1.287 960 0	1.230 444 5
10	0.094 860 0	0.094 860 0	2.2	2.289 825 9	1.998 652 1
11	0.086 672 9	0.085 263 6	2.4	86.431 341 5	8.104 364 1
12	0.079 781 8	0.076 938 1			
14	0.068 828 3	0.063 284 3			
16	0.060 514 2	0.052 687 3			

in the frequency equation, the transformation can be carried out in the same way as before because it does not depend on the particular values represented by the constants in (43). Therefore, our method leads to the following formulae corresponding to (4) and (1) respectively:

$$\frac{2J_0(kb)G_1(0)}{\pi(ka)^2 J_0(ka)F_0(ka)} = \tag{44}$$

$$= -\sum_{n=1,2} \frac{A_n\{\cosh(K_nD/a)[1-(d/2D)] - \cos\beta_0 D\cosh(K_nd/2a)\}}{K_n^2[\cosh(K_nD/a) - \cos\beta_0 D]} G_1\left(\frac{iK_nd}{2a}\right) +$$

$$+ i\sin\beta_0 D \sum_{n=1,2} \frac{A_n \sinh(K_nd/2a)}{K_n^2[\cosh(K_nD/a) - \cos\beta_0 D]} G_2\left(\frac{iK_nd}{2a}\right);$$

$$\frac{2J_0(kb)}{\pi(ka)^2 J_0(ka)F_0(ka)} = \tag{45}$$

$$= -\sum_{n=1,2} \frac{A_n\{\cosh(K_nD/a)[1-(d/2D)] - \cos\beta_0 D\cosh(K_nd/2a)\}}{K_n^2[\cosh(K_nD/a) - \cos\beta_0 D]} I_0\left(\frac{K_nd}{2a}\right),$$

in which

$$K_n = \sqrt{B_n^2 - k^2 a^2}\,, \qquad (n = 1, 2).$$

These results are particularly suited for the calculation of the diameter $2b$ as a function of the other guide dimensions, the frequency and the angle characterizing the mode. They also permit the calculation of the variation of the frequency over the entire pass band as a function of the phase angle $\beta_0 D$ changing from zero (0 mode) to π (π mode) in a given wave guide. In the article following the present paper, the frequencies measured in six irisloaded wave guides have been compared with the values calculated by means of (45), using different sets of constants A_1, A_2, B_1 and B_2.

Without using experimental results, it is also possible to test the accuracy of our method by computing the right hand side of (1) in some special cases. As an example, let us return to the case chosen in the introduction. Taking the factor d/D into account, the sum of the thirteen most important terms of the series in (1) is equal to 0.330 57. The approximation yields the following expression for this series

$$(46) \qquad \frac{J_1(ka)}{J_0(ka)} - ka \sum_{n=1,2} \frac{A_n}{K_n^2} \frac{\cosh(K_n D/a)[1 - (d/2D)] - \cos\beta_0 D \cosh(K_n d/2a)}{\cosh(K_n D/a) - \cos\beta_0 D} I_0\left(\frac{K_n d}{2a}\right).$$

In our example, $\beta_0 D$ is equal to $\pi/2$ so that (46) is somewhat simplified. Making use of the four constants appearing in eq. (42), the expression (46) is found to be 0.33118, which is less than 0.2% larger than the sum of the thirteen terms. Since the latter probably differs no more than a few tenths of a percent from the exact values of the infinite series, the agreement is certainly very satisfactory as it was to be expected.

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Appendix I

Summation of the series

$$f_1(x) \equiv \sum_{m=-\infty}^{\infty} \frac{\cos(m + B)x}{A^2 + (m + B)^2} \qquad (-2\pi \leqslant x \leqslant 2\pi).$$

We can write

$$(47)\qquad f_1(x)=\frac{1}{2iA}\sum_{m=-\infty}^{\infty}(\cos mx\cos Bx-\sin mx\sin Bx)\cdot$$

$$\cdot\left(\frac{1}{m+B-iA}-\frac{1}{m+B+iA}\right)=$$

$$=\frac{\cos Bx}{2iA}\sum_{m=-\infty}^{\infty}\left(\frac{\cos mx}{m+B-iA}-\frac{\cos mx}{m+B+iA}\right)-$$

$$-\frac{\sin Bx}{2iA}\sum_{m=-\infty}^{\infty}\left(\frac{\sin mx}{m+B-iA}-\frac{\sin mx}{m+B+iA}\right).$$

This reduces the problem to the summation of the series

$$\sum_{m=-\infty}^{\infty}\frac{\cos mx}{m+\mu}=\frac{1}{\mu}+2\mu\sum_{m=1}^{\infty}\frac{\cos mx}{\mu^2-m^2}=2\mu\left[\frac{1}{2\mu^2}+\sum_{m=1}^{\infty}\frac{\cos mx}{\mu^2-m^2}\right],$$

$$\sum_{m=-\infty}^{\infty}\frac{\sin mx}{m+\mu}=-2\sum_{m=1}^{\infty}\frac{m\sin mx}{\mu^2-m^2},$$

in which μ is a complex constant. This can be calculated by means of the well-known Fourier series:

$$\cos\mu x=\frac{2\mu\sin\mu\pi}{\pi}\left[\frac{1}{2\mu^2}-\frac{\cos x}{\mu^2-1^2}+\frac{\cos 2x}{\mu^2-2^2}-\ldots\right]\qquad(-\pi\leqslant x\leqslant\pi).$$

Replacing x by $\pi-x$, we get

$$(48)\qquad \cos\mu(\pi-x)=\frac{2\mu\sin\mu\pi}{\pi}\left[\frac{1}{2\mu^2}+\sum_{m=1}^{\infty}\frac{\cos mx}{\mu^2-m^2}\right]\qquad(0\leqslant x\leqslant 2\pi),$$

so that

$$(49)\qquad \sum_{m=-\infty}^{\infty}\frac{\cos mx}{m+\mu}=\frac{\pi\cos\mu(\pi-|x|)}{\sin\mu\pi}\qquad(-2\pi\leqslant x\leqslant 2\pi).$$

By differentiation with respect to x on both sides of (48), we find

$$\sum_{m=1}^{\infty}\frac{m\sin mx}{\mu^2-m^2}=-\frac{\pi\sin\mu(\pi-|x|)}{2\sin\mu\pi}\operatorname{sgn}(x)\qquad(-2\pi<x<2\pi),$$

in which

$$\operatorname{sgn}(x)\equiv\begin{cases}-1 & (x<0)\\ 0 & (x=0)\\ +1 & (x>0).\end{cases}$$

Hence

$$(50)\qquad \sum_{m=-\infty}^{\infty} \frac{\sin mx}{m+\mu} = \frac{\pi \sin \mu(\pi - |x|)}{\sin \mu\pi} \operatorname{sgn}(x) \qquad (-2\pi < x < 2\pi).$$

After having introduced (49) and (50) into (47), we can work out the expressions:

$$f_1(x) = \frac{\pi \cos Bx}{2iA}\left[\frac{\cos (B-iA)(\pi-|x|)}{\sin (B-iA)\pi} - \frac{\cos (B+iA)(\pi-|x|)}{\sin (B+iA)\pi}\right] -$$

$$- \frac{\pi \sin B|x|}{2iA}\left[\frac{\sin (B-iA)(\pi-|x|)}{\sin (B-iA)\pi} - \frac{\sin (B+iA)(\pi-|x|)}{\sin (B+iA)\pi}\right] =$$

$$= \frac{\pi \cos Bx}{2iA}\left[\frac{\cos B(\pi-|x|)\cosh A(\pi-|x|) + i \sin B(\pi-|x|)\sinh A(\pi-|x|)}{\sin B\pi \cosh A\pi - i \cos B\pi \sinh A\pi} -\right.$$

$$\left. - \frac{\cos B(\pi-|x|)\cosh A(\pi-|x|) - i \sin B(\pi-|x|)\sinh A(\pi-|x|)}{\sin B\pi \cosh A\pi + i \cos B\pi \sinh A\pi}\right] -$$

$$- \frac{\pi \sin B|x|}{2iA}\left[\frac{\sin B(\pi-|x|)\cosh A(\pi-|x|) - i \cos B(\pi-|x|)\sinh A(\pi-|x|)}{\sin B\pi \cosh A\pi - i \cos B\pi \sinh A\pi} -\right.$$

$$\left. - \frac{\sin B(\pi-|x|)\cosh A(\pi-|x|) + i \cos B(\pi-|x|)\sinh A(\pi-|x|)}{\sin B\pi \cosh A\pi + i \cos B\pi \sinh A\pi}\right].$$

The fractions with the same denominator can be put together and simplified using the addition theorems for the sine and the cosine function:

$$(51)\qquad f_1(x) = \frac{\pi}{2iA}\left[\frac{\cos B\pi \cosh A(\pi-|x|) + i \sin B\pi \sinh A(\pi-|x|)}{\sin B\pi \cosh A\pi - i \cos B\pi \sinh A\pi} -\right.$$

$$\left. - \frac{\cos B\pi \cosh A(\pi-|x|) - i \sin B\pi \sinh A(\pi-|x|)}{\sin B\pi \cosh A\pi + i \cos B\pi \sinh A\pi}\right] =$$

$$= \frac{\pi}{A} \frac{\cos^2 B\pi \sinh A\pi \cosh A(\pi-|x|) + \sin^2 B\pi \cosh A\pi \sinh A(\pi-|x|)}{\sin^2 B\pi \cosh^2 A\pi + \cos^2 B\pi \sinh^2 A\pi} =$$

$$= \frac{\pi}{A} \frac{\sinh A(2\pi-|x|) + \cos 2B \sinh A|x|}{\cosh 2A\pi - \cos 2B\pi}.$$

This final result is certainly valid in the interval $(-2\pi < x < 2\pi)$.
However, its validity can also be proved for $x = \pm 2\pi$. Indeed, for $x = 0$ we find

$$f_1(0) = \sum_{m=-\infty}^{\infty} \frac{1}{A^2 + (m+B)^2} = \frac{\pi}{A} \frac{\sinh 2A\pi}{\cosh 2A\pi - \cos 2B\pi}.$$

For $x = \pm 2\pi$, f_1 becomes

$$f_1(\pm 2\pi) = \sum_{m=-\infty}^{\infty} \frac{\cos (m+B)2\pi}{A^2 + (m+B)^2} = f_1(0) \cos 2B\pi = \frac{\pi}{A} \frac{\sinh 2A\pi \cos 2B\pi}{\cosh 2A\pi - \cos 2B\pi}.$$

This is exactly the same result as the one obtained by application of (51) when $x = \pm 2\pi$. Hence, (51) is valid in the interval $(-2\pi \leqslant x \leqslant 2\pi)$.

Appendix II

Summation of the series

$$f_2(x) \equiv \sum_{m=-\infty}^{\infty} \frac{\sin (m+B)x}{A^2 + (m+B)^2} \qquad (-2\pi \leqslant x \leqslant 2\pi).$$

We can write:

$$f_2(x) = \frac{1}{2iA} \sum_{m=-\infty}^{\infty} (\sin mx \cos Bx + \cos mx \sin Bx) \cdot$$

$$\cdot \left(\frac{1}{m + B - iA} - \frac{1}{m + B + iA} \right) =$$

$$= \frac{\cos Bx}{2iA} \sum_{m=-\infty}^{\infty} \left(\frac{\sin mx}{m + B - iA} - \frac{\sin mx}{m + B + iA} \right) +$$

$$+ \frac{\sin Bx}{2iA} \sum_{m=-\infty}^{\infty} \left(\frac{\cos mx}{m + B - iA} - \frac{\cos mx}{m + B + iA} \right)$$

and applying the results of Appendix I, we get:

$$(52) \qquad f_2(x) = \frac{\pi \cos Bx \operatorname{sgn}(x)}{2iA} \left[\frac{\sin (B - iA)(\pi - |x|)}{\sin (B - iA)\pi} - \frac{\sin (B + iA)(\pi - |x|)}{\sin (B + iA)\pi} \right] +$$

$$+ \frac{\pi \sin Bx}{2iA} \left[\frac{\cos (B - iA)(\pi - |x|)}{\sin (B - iA)\pi} - \frac{\cos (B + iA)(\pi - |x|)}{\sin (B + iA)\pi} \right] =$$

$$= \frac{\pi \cos Bx \operatorname{sgn}(x)}{2iA} \left[\frac{\sin B(\pi - |x|) \cosh A(\pi - |x|) - i \cos B(\pi - |x|) \sinh A(\pi - |x|)}{\sin B\pi \cosh A\pi - i \cos B\pi \sinh A\pi} - \right.$$

$$\left. - \frac{\sin B(\pi - |x|) \cosh A(\pi - |x|) + i \cos B(\pi - |x|) \sinh A(\pi - |x|)}{\sin B\pi \cosh A\pi + i \cos B\pi \sinh A\pi} \right] +$$

$$+ \frac{\pi \sin B|x| \operatorname{sgn}(x)}{2iA} \left[\frac{\cos B(\pi - |x|) \cosh A(\pi - |x|) + i \sin B(\pi - |x|) \sinh A(\pi - |x|)}{\sin B\pi \cosh A\pi - i \cos B\pi \sinh A\pi} - \right.$$

$$\left. - \frac{\cos B(\pi - |x|) \cosh A(\pi - |x|) - i \sin B(\pi - |x|) \sinh A(\pi - |x|)}{\sin B\pi \cosh A\pi + i \cos B\pi \sinh A\pi} \right] =$$

$$= \frac{\pi \operatorname{sgn}(x)}{2iA} \left[\frac{\sin B\pi \cosh A(\pi - |x|) - i \cos B\pi \sinh A(\pi - |x|)}{\sin B\pi \cosh A\pi - i \cos B\pi \sinh A\pi} - \right.$$

$$\left. - \frac{\sin B\pi \cosh A(\pi - |x|) + i \cos B\pi \sinh A(\pi - |x|)}{\sin B\pi \cosh A\pi + i \cos B\pi \sinh A\pi} \right] =$$

$$= \frac{\pi \operatorname{sgn}(x)}{A} \frac{\sin B\pi \cos B\pi [\sinh A\pi \cosh A(\pi - |x|) - \cosh A\pi \sinh A(\pi - |x|)]}{\sin^2 B\pi \cosh^2 A\pi + \cos^2 B\pi \sinh^2 A\pi} =$$

$$= \frac{\pi}{A} \frac{\sin 2B\pi \sinh Ax}{\cosh 2A\pi - \cos 2B\pi} \qquad (-2\pi \leqslant x \leqslant 2\pi).$$

The validity of this final result in the mentioned closed interval can be proved as in Appendix I.

Appendix III

The problem consists in the determination of the coefficients A_1, A_2, B_1 and B_2 appearing in

$$y = \frac{A_1}{B_1^2 + x^2} + \frac{A_2}{B_2^2 + x^2}$$

as functions of the coordinates (x_α, y_α) $(\alpha = 1, 2, 3, 4)$ of four different points which must be situated on the curve. In other words, we have to solve the system of equations

$$\frac{A_1}{B_1^2 + x_\alpha^2} + \frac{A_2}{B_2^2 + x_\alpha^2} = y_\alpha \qquad (\alpha = 1, 2, 3, 4). \tag{53}$$

Regarding the first two equations as a linear system for A_1 and A_2, the solution is

$$A_1 = \frac{(B_1^2 + x_1^2)(B_1^2 + x_2^2)[y_2(B_2^2 + x_2^2) - y_1(B_2^2 + x_1^2)]}{(B_1^2 - B_2^2)(x_2^2 - x_1^2)},$$

$$A_2 = \frac{(B_2^2 + x_1^2)(B_2^2 + x_2^2)[y_1(B_1^2 + x_1^2) - y_2(B_1^2 + x_2^2)]}{(B_1^2 - B_2^2)(x_2^2 - x_1^2)}.$$

The remaining two equations can be treated in the same way and the solution is directly obtained replacing $x_1 y_1 x_2 y_2$ respectively by $x_3 y_3 x_4 y_4$. Both expressions for A_1 and for A_2 have to be equal; therefore

$$\frac{(B_1^2 + x_1^2)(B_1^2 + x_2^2)[y_2(B_2^2 + x_2^2) - y_1(B_2^2 + x_1^2)]}{(x_2^2 - x_1^2)} =$$

$$= \frac{(B_1^2 + x_3^2)(B_1^2 + x_4^2)[y_4(B_2^2 + x_4^2) - y_3(B_2^2 + x_3^2)]}{(x_4^2 - x_3^2)}, \tag{54}$$

$$(55)\qquad \frac{(B_2^2+x_1^2)(B_2^2+x_2^2)[y_1(B_1^2+x_1^2)-y_2(B_1^2+x_2^2)]}{(x_2^2-x_1^2)} = \frac{(B_2^2+x_3^2)(B_2^2+x_4^2)[y_3(B_1^2+x_3^2)-y_4(B_1^2+x_4^2)]}{(x_4^2-x_3^2)}.$$

These equations are of the third degree in B_1^2 and B_2^2. They can be reduced to second degree equations by suitable combination.

By simple addition, we find after having divided by $B_1^2-B_2^2$ on both sides:

$$(56)\qquad \frac{y_2(B_1^2+x_2^2)(B_2^2+x_2^2)-y_1(B_1^2+x_1^2)(B_2^2+x_1^2)}{(x_2^2-x_1^2)} = \frac{y_4(B_1^2+x_4^2)(B_2^2+x_4^2)-y_3(B_1^2+x_3^2)(B_2^2+x_3^2)}{(x_4^2-x_3^2)}.$$

Multiplying (54) by B_2^2 and (55) by B_1^2, adding them together and dividing by $B_2^2-B_1^2$, we get

$$(57)\qquad \frac{y_2x_1^2(B_1^2+x_2^2)(B_2^2+x_2^2)-y_1x_2^2(B_1^2+x_1^2)(B_2^2+x_1^2)}{(x_2^2-x_1^2)} = \frac{y_4x_3^2(B_1^2+x_4^2)(B_2^2+x_4^2)-y_3x_4^2(B_1^2+x_3^2)(B_2^2+x_3^2)}{(x_4^2-x_3^2)}.$$

Now when we use the substitution

$$P\equiv B_1^2, \qquad Q\equiv B_2^2,$$

(56) and (57) become:

$$(58)\qquad \alpha_1PQ+\beta_1(P+Q)+\gamma_1=0$$

$$(59)\qquad \alpha_2PQ+\beta_2(P+Q)+\gamma_2=0$$

in which we define

$$(60)\qquad \begin{cases} \alpha_1=\dfrac{y_2-y_1}{x_2^2-x_1^2}-\dfrac{y_4-y_3}{x_4^2-x_3^2}, & \alpha_2=\dfrac{y_2x_1^2-y_1x_2^2}{x_2^2-x_1^2}-\dfrac{y_4x_3^2-y_3x_4^2}{x_4^2-x_3^2}, \\ \beta_1=\dfrac{y_2x_2^2-y_1x_1^2}{x_2^2-x_1^2}-\dfrac{y_4x_4^2-y_3x_3^2}{x_4^2-x_3^2}, & \beta_2=\dfrac{(y_2-y_1)x_1^2x_2^2}{x_2^2-x_1^2}-\dfrac{(y_4-y_3)x_3^2x_4^2}{x_4^2-x_3^2}, \\ \gamma_1=\dfrac{y_2x_2^4-y_1x_1^4}{x_2^2-x_1^2}-\dfrac{y_4x_4^4-y_3x_3^4}{x_4^2-x_3^2}, & \gamma_2=\dfrac{(y_2x_2^2-y_1x_1^2)x_1^2x_2^2}{x_2^2-x_1^2}-\dfrac{(y_4x_4^2-y_3x_3^2)x_3^2x_4^2}{x_4^2-x_3^2}. \end{cases}$$

Solving Q from eq. (59) and substituting in eq. (58), we find the following quadratic equation for P:

$$(61)\qquad (\alpha_1\beta_2-\alpha_2\beta_1)P^2+(\alpha_1\gamma_2-\alpha_2\gamma_1)P+(\beta_1\gamma_2-\beta_2\gamma_1)=0$$

or

$$(61') \qquad \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ 1 & -P & P^2 \end{vmatrix} = 0$$

which is a handy form for the computations. The elimination of P leads to the same equation for Q. Therefore the two roots of eq. (61) provide directly the two constants B_1^2 and B_2^2 as functions of the coordinates of the four given points.

In practice, the numerical computations must start with the calculation of the six constants (60). Putting their values into (61′), the solution gives us directly the values of B_1^2 and B_2^2. When the latter are brought into the system (53), we obtain four compatible equations with two unknown A_1 and A_2. The solution of any group of two equations chosen among the four ones leads to the exact values of A_1 and A_2.

RIASSUNTO (*)

L'equazione della frequenza risultante da una teoria approssimata data da WALKINSHAW, è stata generalizzata e trasformata in una nuova formula esatta che contiene ancora una serie infinita. La stessa trasformazione matematica si può applicare all'equazione della frequenza dopo aver introdotto un'opportuna approssimazione per semplificare il termine generale della serie a secondo membro. Si arriva così alla desiderata equazione approssimata in forma chiusa. Poichè l'approssimazione in questione dipende dalla scelta di quattro parametri si discutono i possibili metodi per ottenere una sufficiente esattezza.

(*) *Traduzione a cura della Redazione.*

Experimental Verification of a Frequency Equation for Corrugated Wave Guides.

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(ricevuto il 27 Ottobre 1954)

Summary. — The resonance frequencies measured in several terminated irisloaded wave guides have been compared with theoretical values computed in two different ways by the use of a frequency equation resulting from a previous article. The measured breadth of the lowest pass band in one of the guides has also been compared with the value obtained by means of other formulae and all the cases have been discussed in detail.

In the preceding article (1), it has been shown how the frequency equation

$$\frac{F_1(ka)}{F_0(ka)} = \frac{d}{D} \sum_{m=-\infty}^{\infty} \frac{k}{\chi_m} \frac{J_1(\chi_m a)}{J_0(\chi_m a)} \frac{J_0(\beta_m d/2) \sin(\beta_m d/2)}{\beta_m d/2} \tag{1}$$

which was obtained by Walkinshaw (2) as a result of an approximate theory of periodically irisloaded wave guides, can be replaced to a high degree of accuracy by the approximate formula

$$\frac{2J_0(kb)}{\pi(ka)^2 J_0(ka) F_0(ka)} = -\sum_{n=1,2} \frac{A_n \{\cosh(K_n D/a)[1-(d/2D)] - \cos\beta_0 D \cosh(K_n d/2a)\}}{K_n^2[\cosh(K_n D/a) - \cos\beta_0 D]} I_0(K_n d/2a) \tag{2}$$

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(1) C. C. Grosjean: *Nuovo Cimento*, **12**, 174 (1954). References to this paper will be preceded by the symbol G.

(2) W. Walkinshaw: *Proc. Phys. Soc., London*, **61**, 246 (1948).

which is obviously simpler for numerical computations than (1) containing an infinite series.

We have found it interesting to carry out some experiments in order to compare some frequencies ν_{th} calculated by means of (2) with the corresponding measured values ν_{exp}. To do this, we have constructed a terminated irisloaded guide consisting of a ring symmetrically surrounded by two iris slices and two half cavities, as shown on Fig. 1. Such a system has the advantage that some desired changes in dimension can easily be made. In our experiments we have used three pairs of slices only differing by their iris diameter and the guide diameter $2b$ has only once been enlarged. The dimensions d and D have been kept fixed. All the dimensions have been chosen in the region of those appearing in the design of the linear electron accelerator constructed at the University of Ghent. It is known that in the described systems there exist three resonance modes in the lowest pass band. They are characterized by the phase angle 0, $\pi/2$ and π. In each case, their frequencies have been measured and there is a good reason why it is particularly interesting to compare just those three modes with the theory. Indeed, using the results of measurements which we performed with terminated irisloaded guides containing a variable number of identical cavities, we found that in the lowest pass band the points representing the frequencies plotted vs. $\beta_0 D$ are all lying on a smooth curve which can be very accurately described by

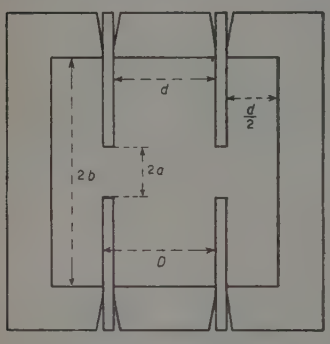

Fig. 1.

$$\nu = A + B\cos\beta_0 D + C\cos^2\beta_0 D \qquad (A \gg B \gg C). \tag{3}$$

Applying this to the cases $\beta_0 D = 0$, $\pi/2$ and π, and solving for A, B and C, we find

$$\nu = \nu_{\pi/2} - \frac{\nu_\pi - \nu_0}{2}\cos\beta_0 D + \frac{\nu_0 - 2\nu_{\pi/2} + \nu_\pi}{2}\cos^2\beta_0 D\,. \tag{4}$$

This seems to hold at least for cavities with dimensions of the order of those which will be mentioned in Table II.

As an example, let us consider the following case:

$D = 25.00$ mm ; $d = 22.50$ mm ; $2a = 25.00$ mm ; $2b = 78.98$ mm [3].

[3] All the dimensions appearing in this article, have been realized within the accuracy of ± 0.005 mm. The experimental error on all the measured frequencies is estimated to be around ± 0.3 MHz.

Using the measured values (in MHz)

$$\nu_0 = 2940.3\,, \qquad \nu_{\pi/2} = 3002.7\,, \qquad \nu_\pi = 3060.8\,,$$

we find

$$\nu = 3002.70 - 60.25\cos\beta_0 D - 2.15\cos^2\beta_0 D \tag{5}$$

and in Table I, a large number of measured frequencies ν_{exp} are compared with the values ν_c calculated by means of (5). All the frequency determinations

TABLE I.

$\beta_0 D/\pi$	ν_{exp}	ν_c	$\beta_0 D/\pi$	ν_{exp}	ν_c	$\beta_0 D/\pi$	ν_{exp}	ν_c
0	2940.3	2940.3	1/3	2971.9	2972.0	7/10	3037.4	3037.4
1/16	2941.5	2941.5	5/14	2976.2	2976.2	5/7	3039.6	3039.4
1/15	2941.6	2941.7	3/8	2979.3	2979.3	11/15	3042.3	3042.1
1/14	2942.0	2941.9	2/5	2984.0	2983.9	3/4	3044.3	3044.2
1/12	2942.6	2942.5	5/12	2987.2	2987.0	7/9	3048.0	3047.6
1/10	2943.6	2943.5	3/7	2989.3	2989.2	11/14	3048.7	3048.5
1/9	2944.2	2944.2	7/16	2991.1	2990.9	4/5	3050.0	3050.0
1/8	2945.4	2945.2	4/9	2992.1	2992.2	13/16	3051.3	3051.3
2/15	2945.7	2945.9	7/15	2996.4	2996.4	5/6	3052.9	3053.3
1/7	2946.6	2946.7	1/2	3002.7	3002.7	6/7	3054.9	3055.2
1/6	2949.0	2948.9	8/15	3009.0	3009.0	13/15	3056.2	3056.0
3/16	2951.2	2951.1	5/9	3013.1	3013.1	7/8	3056.5	3056.5
1/5	2952.6	2952.5	9/16	3014.4	3014.4	8/9	3057.4	3057.4
3/14	2954.4	2954.3	4/7	3016.0	3016.0	9/10	3057.9	3058.1
2/9	2955.5	2955.3	7/12	3018.2	3018.1	11/12	3058.8	3058.9
1/4	2958.9	2959.0	3/5	3021.0	3021.1	13/14	3059.5	3059.4
4/15	2961.2	2961.4	5/8	3025.6	3025.4	14/15	3059.6	3059.6
2/7	2964.2	2964.3	9/14	3028.3	3028.4	15/16	3059.7	3057.7
3/10	2966.5	2966.5	2/3	3032.1	3032.3	1	3060.8	3060.8
5/16	2968.4	2968.6	11/16	3035.6	3035.5			

were carried out by comparison with a wave meter standardized for vacuum. Since both, cavities and wave meter, were filled with the same air, it is clear that the measurements provided directly the resonance frequencies for the terminated guides in vacuum.

The excellent agreement between the experimental and the computed values in Table I illustrates sufficiently well the importance of the three modes whose frequencies we need for the calculation of eq. (4). Therefore, it should be examined in how far eq. (2) can be used to compute these three resonance frequencies.

First, all the calculations have been made with a fixed choice for the constants A_1, A_2, B_1 and B_2, namely with the values mentioned in the preceding article (G 42):

$$\begin{cases} A_1 = 2.2390948\,, \\ A_2 = 17.1548583 \end{cases} \qquad \begin{cases} B_1^2 = 6.0409706\,, \\ B_2^2 = 132.6251039\,. \end{cases} \tag{6}$$

For the 0, $\pi/2$ and π modes, the right hand side of eq. (2) can be somewhat simplified and we get respectively:

$$(7)\qquad \begin{cases} -\sum_{n=1,2} \dfrac{A_n}{K_n^2} \dfrac{\sinh[K_n(D-d)/2a]}{\sinh(K_nD/2a)} I_0(K_nd/2a)\,, \\ -\sum_{n=1,2} \dfrac{A_n}{K_n^2} \dfrac{\cosh(K_nD/a)[1-(d/2D)]}{\cosh(K_nD/a)} I_0(K_nd/2a)\,, \\ -\sum_{n=1,2} \dfrac{A_n}{K_n^2} \dfrac{\cosh[K_n(D-d)/2a]}{\cosh(K_nD/2a)} I_0(K_nd/2a)\,. \end{cases}$$

In our examples, the factors K_2, $K_2d/2a$ and $K_2D/2a$ were of the order of 10, so that to a high degree of approximation, the hyperbolic functions in the denominators could be replaced by exponentials. The second terms of the above expressions become respectively

$$(8)\qquad -\frac{A_2}{K_2^2} I_0\left(\frac{K_2d}{2a}\right) \exp\left[-\frac{K_2d}{2a}\right] \cdot \begin{cases} 1-\exp\left[-\dfrac{K_2(D-d)}{a}\right] & (0\text{ — mode}) \\ 1 & (\pi/2\text{ — mode}) \\ 1+\exp\left[-\dfrac{K_2(D-d)}{a}\right] & (\pi\text{ — mode}) \end{cases}$$

and this was an interesting simplification since we disposed of a table of the function $I_0(x)e^{-x}$.

Our final results are contained in Table II. The difference $\Delta\nu=\nu_{th}-\nu_{exp}$ turns out positive in all the cases, and increases steadily with increasing β_0D, increasing radius a and decreasing radius b. The relative errors are resp. about 0.5, 0.7 and 1.1% for the 0, $\pi/2$ and π modes. This makes the relative deviation between the measured and the calculated breadth of the lowest pass bands rather large as it will be shown in Table IV. The mentioned deviations for the different modes are expected to be caused partially by the inaccuracy of Walkinshaw's frequency equation and partially by the use of the constants (6) in the approximate eq. (2). In order to have an idea about the order of magnitude of both inaccuracies, we have applied equation (2) for one of the constructed systems of cavities, recalculating the constants A_1, A_2, B_1 and B_2 for each mode as it was pointed out in the previous article. We obtained the results given in Table III. Each of the constants (6) is lying between the corresponding constants of the first two sets in Table III. Hence, it is not surprising that the results obtained with the set (6) are practically the same as those of Table III for the 0 and $\pi/2$ mode. At the same time, this proves that at least within certain limits the calculated frequencies are rather insensitive to the choice of the constants A_1, A_2, B_1 and B_2. However, in the case of the π mode, the differences between the set (6) and the set mentioned in

TABLE II.

$D = 25.00$ mm $\quad d = 22.50$ mm

$2b$ (mm)	$2a$ (mm)	$\beta_0 D$	ν_{exp}	ν_{th}	$\Delta\nu$
75.00	23.00	0	3 094.8	3 109.7	14.9
		$\pi/2$	3 148.9	3 170.6	21.7
		π	3 201.5	3 233.1	31.6
	25.00	0	3 100.6	3 116.5	15.9
		$\pi/2$	3 172.7	3 195.1	22.4
		π	3 240.0	3 276.2	36.2
	27.00	0	3 106.5	3 123.1	16.6
		$\pi/2$	3 198.5	3 222.4	23.9
		π	3 285.3	3 325.4	40.1
79.00	23.00	0	2 934.7	2 948.4	13.7
		$\pi/2$	2 980.3	3 000.8	20.5
		π	3 025.3	3 054.1	28.8
	25.00	0	2 939.7	2 954.5	14.8
		$\pi/2$	3 001.4	3 022.2	20.8
		π	3 059.3	3 091.3	32.0
	27.00	0	2 945.1	2 960.2	15.1
		$\pi/2$	3 023.6	3 046.0	22.4
		π	3 098.7	3 133.7	35.0

TABLE III.

$D = 25.00$ mm $\quad d = 22.50$ mm $\quad 2a = 25.00$ mm $\quad 2b = 75.00$ mm

$\beta_0 D$	Constants	ν_{exp}	ν_{th}	$\Delta\nu$	% dev.
0	$A_1 = 2.570\,704\,3$ $A_2 = 19.489\,712\,1$ $B_1^2 = 6.558\,296\,2$ $B_2^2 = 176.930\,317\,8$	3 100.6	3 116.3	15.7	0.51
$\pi/2$	$A_1 = 2.232\,992\,0$ $A_2 = 14.125\,506\,3$ $B_1^2 = 6.064\,540\,3$ $B_2^2 = 107.137\,322\,3$	3 172.7	3 195.3	22.6	0.71
π	$A_1 = 3.077\,072\,7$ $A_2 = 23.417\,396\,2$ $B_1^2 = 7.778\,326\,0$ $B_2^2 = 252.082\,018\,0$	3 240.0	3 253.4	13.4	0.41

Table III become really too large, so that the calculated frequencies differ considerably. Comparing the corresponding frequencies in the Tables II and III, it is clear that the set of constants (6) is very well chosen for the calculation of 0 and $\pi/2$ modes, but not for π modes. It would probably be difficult to find a set of constants which would lead to the highest precision for all the modes at once. All these remarks are closely connected to the discussion about the choice of the approximation made in the preceding article.

It is highly probable that the remaining differences between theory and experiment, namely the values of $\Delta\nu$ given in Table III, are largely due to the inaccuracy of Walkinshaw's choice (G 2) for the field component E_z at the corrugation mouths. In connection with this statement, we can ask ourselves why $\Delta\nu$ is somewhat larger for the $\pi/2$ mode than for the other modes. There are several possible explanations for this larger discrepancy:

1) the field at the corrugation mouths chosen by WALKINSHAW, may be a better approximation for the 0 and the π mode than for the $\pi/2$ mode;

2) in the preceding paper, it was shown that by the method used to calculate the values given in Table III, one takes resp. seven, four and eight terms exactly into account in the series of the right hand side of (1) for the considered modes. Our approximation method may therefore be slightly less accurate for the $\pi/2$ mode;

3) it should be remembered that in general the function representing the field component E_z at the corrugation mouth must be (apart from an arbitrary phase factor) the sum of a symmetric real part and an antisymmetric imaginary part. Now, Walkinshaw's trial function (G 2) contains only an even part. However, for the 0 and the π mode, the omission of the odd part does not at all affect the results, because in the exact frequency equation (G 34) all the terms depending on the odd part of E_z contain the factor $\sin \beta_0 D$ which vanishes for the mentioned modes. This is not the case when $0 < \beta_0 D < \pi$. Here, the contribution of the terms in (G 34) containing the odd part of E_z is not a priori equal to zero and the omission of similar terms in (G 44) may introduce an additional inaccuracy for the $\pi/2$ mode.

Finally, let us consider the breadth of the lowest pass band. In Table IV, the experimental values for one set of cavities are compared with the following three approximations:

— column (A) contains the frequencies calculated by means of the fixed constants (6);

— column (B) contains the frequencies computed with suitably chosen values of the constants, already mentioned in Table III,

— column (C) gives the frequencies calculated by the use of formulae derived by COMBE (4).

TABLE IV.

$D = 25.00$ mm $d = 22.50$ mm $2a = 25.00$ mm $2b = 75.00$ mm				
	exp.	(A)	(B)	(C)
ν_0	3100.6	3116.5	3116.3	3110.0
ν_π	3240.0	3276.2	3253.4	3332.9
$\nu_\pi - \nu_0$	139.4	159.7	137.1	222.9
% dev.	—	14.6	1.6	59.9

In the last row, we mention the percentage deviation from the experimental band width. For the other cavity systems appearing in Table II, the deviations remain of the same order of magnitude. The result of column (B) is remarkably better than the other ones. Calculated with Combe's formulae, ν_0 turns out 0.2% more accurate than our values, whereas the absolute error on ν_π is about seven times as large as on our result in (B), producing an enormous percentage deviation on the band width. This can be rather well explained in the following way.

In the general case of a 0 mode ($\beta_0 = 0$) and thin irisses [$(d/D) \cong 1$], the replacement of the series in (1) by the term $m = 0$ can be justified in some sense. Indeed, for practically all values of m different from zero, we can write

$$\frac{\sin(\beta_m d/2)}{\beta_m d/2} = \frac{\sin(m\pi d/D)}{m\pi d/D} \ll 1 ,$$

and we must take into account that most of the time, the factor

$$\frac{J_1(\chi_m a)}{\chi_m a J_0(\chi_m a)}$$

decreases fairly rapidly with increasing m. Actually, when d is equal to D (infinitely thin irisses), the frequency equation reduces exactly to

$$\frac{F_1(ka)}{F_0(ka)} = \frac{J_1(ka)}{J_0(ka)} , \tag{9}$$

which we can transform into

$$\frac{2J_0(kb)}{\pi ka J_0(ka) F_0(ka)} = 0 \tag{10}$$

(4) R. COMBE: *Compt. Rend. Paris*, **238**, 1697 (Avril 1954).

using the well-known relation (G 33). This proves that in the limiting case $d = D$, the right hand side of (1) reduces to a single term and the obtained frequency equation which is equivalent to

$$J_0(kb) = 0 \tag{11}$$

becomes at the same time exact. The case in which the iris thickness is smali compared to D can be regarded as some sort of a perturbation on the preceding case. Such a reasoning does not hold for π modes.

In COMBE's examples the percentage deviation between theory and experiment are a lot smaller than in our example (C) of Table IV. This is probably due to the fact that for his wave guides the ratio a/D is much larger than in our case. This makes the intervals separating the consecutive values of $\chi_m a$ ($m = 0, \pm 1, ...$) larger and therefore improves the convergence of the series in (1).

It is a pleasure to thank the Institut Interuniversitaire des Sciences Nucléaires and Prof. Dr. J. L. VERHAEGHE for his interest in this work.

RIASSUNTO (*)

Le frequenze di risonanza misurate in alcune guide d'onda con diaframmi ad iris finite si confrontano coi valori teorici calcolati in due modi differenti per mezzo di una equazione della frequenza risultante da un precedente lavoro. Anche l'ampiezza misurata della minima banda di frequenza passante in una delle guide si confronta col valore ottenuto per mezzo di altre formule e tutti i casi si discutono dettagliatamente.

(*) *Traduzione a cura della Redazione.*

Remarks on Time Reversal.

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Summary. — A rigorous proof is given for the irreversibility of a system consisting of scalar neutral mesons coupled to a «nucleon» field and already discussed in a previous paper (1).

In a previous paper (hereafter referred to as I) we have shown that in order to decide whether or not a given quantum-mechanical system is reversible one has in general to explore an entire class of Hamiltonians. In particle mechanics this class consists of all the Hamiltonians

$$H' = \exp[iF(\boldsymbol{x})]H \exp[-iF(\boldsymbol{x})], \tag{1}$$

where $F(\boldsymbol{x})$ is an arbitrary function of the position coordinates of the particles. The reason for this is that the canonical momentum $\boldsymbol{p}$ of a particle has no precise physical meaning since an arbitrary $\partial F/\partial \boldsymbol{x}$ may be added to it.

We have shown that to exclude irreversible particle systems it is sufficient to postulate $\hat{\boldsymbol{x}} = \boldsymbol{x}$ in particle mechanics.

We have made a similar postulate:

$$\hat{n}^i_{\boldsymbol{k},s} = n^i_{-\boldsymbol{k},s}, \tag{2}$$

($n^i_{\boldsymbol{k},s}$ = occupation numbers) in the case of fields, which, to decide on the reversibility of a system, forces us to explore the class of Hamiltonians

$$H' = \exp[iF(\mathcal{H})]H \exp[-iF(\mathcal{H})], \tag{3}$$

(1) G. Morpurgo, L. A. Radicati and B. F. Touschek: *Nuovo Cimento*, **12**, 677 (1954).

where F is an arbitrary function of the occupation numbers. It is probable — as mentioned in I, 3.7 — that a full use of the correspondence principle, i.e. application to the wave limit instead of to the particle-limit from which equation (2) results may impose some further limitations on F.

However, even with the « wide » postulate (2) it was possible to show the irreversibility of a system of scalar neutral mesons coupled to a nucleon field by a scalar and vector coupling term. It was shown that no function F exists in this case. The importance of this example, which is of no great physical interest, lies in the fact that it represents a non trivial case of a time irreversible system of fields. In view of this importance we want to present here a rigorous proof for the non existence of the function F in this case, particularly since we found that the proof presented in I was in some points lacking mathematical rigour.

The argument on which the following proof will be based may be more generally useful for a larger class of problems, as for example that of Fermion types, with we hope to discuss in another paper.

Let $n_{\boldsymbol{k}}$ be the number of Bosons with momentum $\boldsymbol{k}$ and $n_{\boldsymbol{p},s,\varepsilon}$ the number of Fermions in the state $\boldsymbol{p}$, s, ε. Then a generic function $F(n_{\boldsymbol{k}_1}, \ldots, n_{\boldsymbol{k}}, \ldots; n_{\boldsymbol{p}_1,s_1,\varepsilon_1}, \ldots, n_{\boldsymbol{p},s,\varepsilon}, \ldots)$ can always be represented in the form

$$(4) \qquad F(n_{\boldsymbol{k}_1}, \ldots, n_{\boldsymbol{k}}, \ldots; n_{\boldsymbol{p}_1,s_1,\varepsilon}, \ldots n_{\boldsymbol{p},s,\varepsilon} \ldots) = \alpha(\ldots n_{\boldsymbol{k}} \ldots) +$$

$$+ \sum_{\boldsymbol{p},s,\varepsilon} \alpha(\boldsymbol{p}, s, \varepsilon \,|\, \ldots n_{\boldsymbol{k}} \ldots) n_{\boldsymbol{p},s,\varepsilon} +$$

$$+ \sum_{\boldsymbol{p},s,\varepsilon \neq \boldsymbol{p}',s',\varepsilon'} \alpha(\boldsymbol{p}, s, \varepsilon; \boldsymbol{p}', s', \varepsilon' \,|\, \ldots n_{\boldsymbol{k}} \ldots) n_{\boldsymbol{p},s,s} \cdot n_{\boldsymbol{p}',s',\varepsilon'} + \ldots .$$

This expansion differs from that used in I in that it is an expansion with respect to the numbers of Fermions only; if one would also expand with respect to the Boson occupation numbers $n_{\boldsymbol{k}}$ the coefficients of this expansion would not be unique. In the second term the contributions $\boldsymbol{p}, s, \varepsilon = \boldsymbol{p}', s', \varepsilon'$ are excluded since — owing to the relation $n^2_{\boldsymbol{p},s,\varepsilon} = n_{\boldsymbol{p},s,\varepsilon}$ — they appear already in the linear term of the expansion.

That (4) is the most general representation of the function F can be seen in the following manner: assume that there are N states available to the individual Fermion. The total number of Fermion states is then 2^N and the series (4) will break off with the N-th term. The number of coefficients α is

$$1 + \binom{N}{1} + \binom{N}{2} + \ldots + \binom{N}{N} = 2^N .$$

They are uniquely determined from (4) by means of the relations

$$\alpha(\ldots n_{\boldsymbol{k}} \ldots) = F(\ldots n_{\boldsymbol{k}} \ldots; 0, 0, \ldots)$$

$$\alpha(\boldsymbol{p}, s, \varepsilon \,|\, \ldots n_{\boldsymbol{k}} \ldots) = F(\ldots n_{\boldsymbol{k}} \ldots; 0, 0, \ldots 1_{\boldsymbol{p},s,\varepsilon}, 0, \ldots), \quad \text{etc.}$$

We now want to go back to equation (86) of I for which we may write

$$F(n'') - F(n') = {}_{n''}\Gamma_{n'}, \tag{5}$$

where Γ is defined by ${}_{n''}R_{n'} = \exp[i\,{}_{n''}\Gamma_{n'}]$. We now consider a process in which one of the n_{k_1} mesons of momentum $\boldsymbol{k}_1$ is absorbed (in the presence of an arbitrary number of other mesons) by a nucleon which makes a transition from the state $\boldsymbol{p}$, s, ε to the state $\boldsymbol{p}+\boldsymbol{k}_1$, s, ε. We restrict the following consideration to those transitions in which the spin and the sign of energy remain unchanged and in which in the initial state only the nucleon $\boldsymbol{p}$, s, ε is present. Then suspressing the spin and energy indices (which are $++$ throughout) we have for (5)

$$\alpha'(\boldsymbol{p}\,|\ldots n_{\boldsymbol{k}}\ldots n_{\boldsymbol{k}_1}\ldots) = \alpha'(\boldsymbol{p}+\boldsymbol{k}_1\,|\ldots n_{\boldsymbol{k}}\ldots n_{\boldsymbol{k}_1}-1\ldots) + \gamma(\boldsymbol{p}_1\boldsymbol{k}_1) \tag{6}$$

where we have put

$${}_{n''}\Gamma_{n'} = \Gamma^{++}_{++}(\boldsymbol{p}+\boldsymbol{k}_1,\,\boldsymbol{p}) = \gamma(\boldsymbol{p}_1\boldsymbol{k}_1) \tag{7}$$

and

$$\alpha'(\boldsymbol{p}\,|\ldots n_{\boldsymbol{k}}\ldots) = \alpha(\ldots n_{\boldsymbol{k}}\ldots) + \alpha(\boldsymbol{p}\,|\ldots n_{\boldsymbol{k}}\ldots)\,. \tag{8}$$

Higher terms in the expansion (4) do not contribute since in either initial and final state only one Fermion state is occupied.

We want to show that the system of equations (6) has no solutions, if Γ has the form derived in the previous paper (compare equation (88) and the equation following (89)). In order to show this assume that the system of equations (6) has a solution. It then follows from equation (6) by iteration (reducing in each step the number $n_{\boldsymbol{k}}$ by one), that

$$\alpha'(\boldsymbol{p}\,|\ldots n_{\boldsymbol{k}}\ldots n_{\boldsymbol{k}_1}\ldots) = \alpha'(\boldsymbol{p}+n_{\boldsymbol{k}_1}\boldsymbol{k}_1\,|\ldots n_{\boldsymbol{k}}\ldots 0_{\boldsymbol{k}_1}\ldots) + \sum_{\nu=0}^{n_{\boldsymbol{k}_1}-1}\gamma(\boldsymbol{p}+\nu\boldsymbol{k}_1,\,\boldsymbol{k}_1)\,. \tag{9}$$

This equation allows one to express the values of α' calculated for a certain occupation number $n_{\boldsymbol{k}_1}$ in terms of the α' calculated for $n_{\boldsymbol{k}_1} = 0$, all the other occupation numbers remaining unaltered. If equations (6) have a solution equation (9) holds generally for every set of $n_{\boldsymbol{k}}$, any $n_{\boldsymbol{k}_1}$ and every value of $\boldsymbol{k}_1$.

Equation (9) can be used to express $\alpha'(\boldsymbol{p}\,|\ldots n_{\boldsymbol{k}_1}\ldots n_{\boldsymbol{k}_2}\ldots)$ in terms of $\alpha'(\boldsymbol{p}+n_{\boldsymbol{k}_1}\boldsymbol{k}_1+n_{\boldsymbol{k}_2}\boldsymbol{k}_2\,|\ldots 0_{\boldsymbol{k}_1}\ldots 0_{\boldsymbol{k}_2}\ldots)$. This can be done in two ways: 1) we first express $\alpha'(\boldsymbol{p}\,|\ldots n_{\boldsymbol{k}_1}\ldots n_{\boldsymbol{k}_2}\ldots)$ in terms of $\alpha'(\boldsymbol{p}+n_{\boldsymbol{k}_1}\boldsymbol{k}_1\,|\ldots 0_{\boldsymbol{k}_1}\ldots n_{\boldsymbol{k}_2}\ldots)$ by using equation (9); then, again by using equation (9) we express $\alpha'(\boldsymbol{p}+n_{\boldsymbol{k}_1}\boldsymbol{k}_1\,|\ldots 0_{\boldsymbol{k}_1}\ldots n_{\boldsymbol{k}_2}\ldots)$

by $\alpha'(\boldsymbol{p} \pm n_{\boldsymbol{k}_1}\boldsymbol{k}_1 + n_{\boldsymbol{k}_2}\boldsymbol{k}_2 | \ldots 0_{\boldsymbol{k}_1} \ldots 0_{\boldsymbol{k}_2} \ldots)$. 2) We first reduce $n_{\boldsymbol{k}_2}$ to 0 and then $n_{\boldsymbol{k}_1}$, always by using (9). If the set of equations (6) admits a solution both methods must lead to the same result. Now, method 1) gives

$$(9.1) \qquad \alpha'(\boldsymbol{p} | \ldots n_{\boldsymbol{k}_1} \ldots n_{\boldsymbol{k}_2} \ldots) = \alpha'(\boldsymbol{p} + n_{\boldsymbol{k}_1}\boldsymbol{k}_1 + n_{\boldsymbol{k}_2}\boldsymbol{k}_2 | \ldots 0_{\boldsymbol{k}_1} \ldots 0_{\boldsymbol{k}_2} \ldots) + \\ + \sum_{\nu=1}^{n_{\boldsymbol{k}_1}-1} \gamma(\boldsymbol{p} + \nu\boldsymbol{k}_1, \boldsymbol{k}_1) + \sum_{\nu=1}^{n_{\boldsymbol{k}_2}-1} \gamma(\boldsymbol{p} + n_{\boldsymbol{k}_1}\boldsymbol{k}_1 + \nu\boldsymbol{k}_2, \boldsymbol{k}_2) ,$$

and method 2)

$$(9.2) \qquad \alpha'(\boldsymbol{p} | \ldots n_{\boldsymbol{k}_1} \ldots n_{\boldsymbol{k}_2} \ldots) = \alpha'(\boldsymbol{p} + n_{\boldsymbol{k}_1}\boldsymbol{k}_1 + n_{\boldsymbol{k}_2}\boldsymbol{k}_2 | \ldots 0_{\boldsymbol{k}_1} \ldots 0_{\boldsymbol{k}_2} \ldots) + \\ + \sum_{\nu=0}^{n_{\boldsymbol{k}_2}-1} \gamma(\boldsymbol{p} + \nu\boldsymbol{k}_2, \boldsymbol{k}_2) + \sum_{\nu=0}^{n_{\boldsymbol{k}_1}-1} \gamma(\boldsymbol{p} + n_{\boldsymbol{k}_2}\boldsymbol{k}_2 + \nu\boldsymbol{k}_1, \boldsymbol{k}_2) ,$$

so that for equation (6) to have a solution the two right hand sides have to be the same. That this cannot be the case for the given example already follows considering only the special case $n_{\boldsymbol{k}_1} = n_{\boldsymbol{k}_2} = 1$. For these values of $n_{\boldsymbol{k}_1}$ and $n_{\boldsymbol{k}_2}$ we obtain equating expressions (9.1) and (9.2)

$$(10) \qquad \gamma(\boldsymbol{p}, \boldsymbol{k}_1) - \gamma(\boldsymbol{p} + \boldsymbol{k}_2, \boldsymbol{k}_1) = \gamma(\boldsymbol{p}, \boldsymbol{k}_2) - \gamma(\boldsymbol{p} + \boldsymbol{k}_1, \boldsymbol{k}_2) .$$

Putting $\boldsymbol{k}_1 \to 0$ (remembering that γ represented by relation (88) of the previous paper is an analytical function of its arguments) we find

$$(11) \qquad \gamma(\boldsymbol{p}, 0) - \gamma(\boldsymbol{p} + \boldsymbol{k}_2, 0) = 0 ,$$

i.e. $\gamma(\boldsymbol{p}, 0) = \Gamma(\boldsymbol{p}, \boldsymbol{p})$ independent of $\boldsymbol{p}$. Since this (because of equation (88) of the previous paper) is not the case, equation (6) can have no solutions and the system under consideration is irreversible.

RIASSUNTO

Si dà una dimostrazione rigorosa della irreversibilità di un sistema, già discusso in un precedente lavoro, formato da mesoni scalari neutri accoppiati a un campo « nucleonico » tramite termini di accoppiamento scalari e vettoriali.

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Zur Formulierung quantisierter Feldtheorien.

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Summary. - A new formulation of quantized field theories is proposed. Starting from some general requirements we derive a set of equations which determine the matrix-elements of field operators and the S-Matrix. These equations contain no renormalization constants, but only experimental masses and coupling parameters. The main advantage over the conventional formulation is thus the elimination of all divergent terms in the basic equations. This means that no renormalization problem arises. The formulation is here restricted to theories which do not involve stable bound states. For simplicity we derive the equations for spin 0 particles, however the extension to other cases (e.g. quantum electrodynamics) is obvious. The solutions of the equations are discussed in a power-series expansion. They are then identical with the renormalized expressions of the conventional formulation. However, the equations set up here are not restricted to the application of perturbation theory.

1. – Einleitung.

Um die Problemstellung dieser Arbeit zu verdeutlichen, sei zunächst auf einige unbefriedigende Züge der gebräuchlichen Formulierung von Feldtheorien hingewiesen:

1) Während es möglich ist, für die sog. renormierbaren Theorien im Rahmen einer Entwicklung nach Kopplungsparametern endliche Resultate für die Matrixelemente der (renormierten) Feldoperatoren und der S-Matrix zu erhalten, ist es bisher nicht gelungen, die Grundgleichungen derartiger Theorien konvergent zu formulieren. Wir meinen damit die Tatsache, daß sowohl in den Feldgleichungen als auch in den Vertauschungsrelationen der renormierten Operatoren divergente Renormierungskonstanten auftreten.

2) Die erwähnten endlichen Resultate lassen sich nur über divergente Zwischenrechnungen gewinnen, die im Zuge der Renormierung vorzunehmen sind. Wenn auch die zur Abspaltung dieser divergenten Terme notwendigen Vorschriften eindeutig formuliert werden können, so sind sie doch eng an die Störungsrechnung gebunden.

Diese Umstände wirken sich sowohl bei der Behandlung prinzipieller Fragen als auch bei der Anwendung der Theorie nachteilig aus. Einmal erschweren sie eine Beantwortung der Frage nach der Existenz von Lösungen der Grundgleichungen (etwa für die Quantenelektrodynamik), zum anderen steht die notwendige Abspaltung (divergenter) Renormierungsglieder einer konsequenten Anwendung anderer Näherungsverfahren als der Entwicklung nach einer Kopplungskonstanten im Wege.

Der von uns im folgenden unternommene Versuch einer Neuformulierung quantisierter Feldtheorien verfolgt das Ziel, die eben geschilderten Schwierigkeiten zu vermeiden. Wesentliches Merkmal dieser Formulierung ist, daß wir Gleichungen ableiten, die im Gegensatz zu den gebräuchlichen Grundgleichungen divergenzfrei sind. Sie reichen im Prinzip aus zur Bestimmung der Matrixelemente der Feldoperatoren und der S-Matrix. Die physikalisch bedeutungslosen Renormierungskonstanten sind vollständig aus der Theorie eliminiert; d.h. sowohl in den Grundgleichungen als auch in allen weiteren Beziehungen kommen nur renormierte Feldoperatoren sowie experimentelle Massen und Kopplungsparameter vor, so daß es kein Renormierungsproblem gibt. Feldgleichungen und kanonische Vertauschungsrelationen der Operatoren werden nicht benutzt. Dieser Verzicht erscheint unerläßlich, wenn man nur mit renormierten Größen arbeiten will.

Es ist hervorzuheben, daß wir eine Neuformulierung, jedoch keine Abänderung der physikalischen Grundlagen der Quantenfeldtheorie im Auge haben. Dies äußert sich darin, daß die störungstheoretischen Lösungen unserer Gleichungen identisch sind mit den entsprechenden renormierten Ausdrücken der üblichen Formulierung. Bei der Ableitung des Gleichungssystems wird nicht vorausgesetzt, daß die Lösungen sich nach einem Kopplungsparameter entwickeln lassen. Insofern bedeutet eine eventuell divergierende störungstheoretische Reihe hier keine prinzipielle Schwierigkeit für die Diskussion der Frage, ob exakte Lösungen der Grundgleichungen existieren.

Wir beschränken uns vorläufig auf die Formulierung von Theorien, in denen keine stabilen gebundenen Teilchen auftreten. Dies sollte z.B. für die Quantenelektrodynamik ausreichen. Um möglichst einfache Verhältnisse zu haben, geben wir die Gleichungen für den Fall an, daß nur eine Teilchensorte vorhanden ist; nämlich Teilchen vom Spin 0 und der Masse m, deren Wechselwirkung zu beschreiben ist. Die Entwicklung läßt sich jedoch unmittelbar auf andere Fälle (z.B. die Quantenelektrodynamik) übertragen.

Für die Ableitung der von uns benutzten Gleichungen sind lediglich einige

sehr allgemeine Voraussetzungen notwendig. Die Gleichungen enthalten dementsprechend viele Lösungen; die invarianten Lösungen entsprechen lokalen Feldtheorien. Die einzelnen Möglichkeiten der Wechselwirkung werden durch Randbedingungen charakterisiert, die bei der Auflösung der Gleichungen zu stellen sind. Diese Auflösung wird vorläufig nur im Rahmen der Störungsrechnung betrachtet. Auf die Behandlung nichtlokaler Felder und auf Fragen, die mit nichtrenormierbaren Theorien zusammenhängen, wird nicht eingegangen.

2. – Allgemeines.

Wir beginnen mit der Darlegung der allgemeinen Grundlagen. Unsere Absicht ist, eine quantentheoretische Formulierung der lokalen, skalaren Felder ohne gebundene Zustände zu geben, die sich von den bekannten Formulierungen darin auszeichnet, daß sie die Schwierigkeiten der Ultraviolettdivergenzen grundsätzlich vermeidet. D.h.: Bereits die Grundgleichungen der Theorie selbst sollen frei von Ultraviolettdivergenzen sein. Es ist klar, daß wir zu diesem Zweck so geläufige methodische Hilfsmittel, wie die Hamiltonfunktion kanonischer Feldvariablen, die kanonischen Vertauschungsrelationen oder das Feynman-Schwingersche Variationsprinzip nicht brauchen können, da diese alle divergente Renormierungskonstanten enthalten. Wir müssen uns also nach noch allgemeineren Eigenschaften quantisierter Felder umsehen, die von solchen Divergenzschwierigkeiten unbelastet und deshalb zur Grundlegung unserer Theorie geeignet sind. Dazu zählt die Existenz einer invarianten, unitären S-Matrix, deren Matrixelemente als beobachtbare Größen zwangsläufig endlich sein müssen. Hierdurch wäre aber erst der Rahmen allgemeiner quantisierter Felder abgesteckt. Zur Kennzeichnung der lokalen Felder muß man darüber hinaus in geeigneter Weise die Forderung der Kausalität stellen. Da es bisher nicht gelungen ist, die Kausalität als Eigenschaft der S-Matrix selbst zu formulieren, entschließen wir uns zur Einführung eines Feldoperators und gehen damit wesentlich über die Konzeption einer reinen S-Matrixtheorie hinaus.

Wir denken uns ein Feld durch einen linearen, hermitischen Operator $A(x)$ des Hilbertraums beschrieben. Diesen « Feldoperator » $A(x)$ (er entspricht genau dem sog. « renormierten Feldoperator » früherer Formulierungen) gilt es nun in seinen Eigenschaften weiter einzuschränken. Das geschieht, indem wir an ihn die drei folgenden Forderungen stellen:

1) Invarianzprinzip. Die Theorie soll invariant sein gegenüber Lorentztransformationen, Translationen, sowie Spiegelungen in Raum und Zeit.

2) Kausalitätsforderung. Der raumartige Kommutator des Feldoperators

soll verschwinden:

$$[A(x), A(y)] = 0\,, \tag{1}$$

wenn $x - y$ ein raumartiger Vektor ist.

3) Asymptotenbedingung. Der Feldoperator $A(x)$ soll für $x_0 \to -\infty$ und $x_0 \to +\infty$ in die wechselwirkungsfreien Feldoperatoren $A_{in}(x)$ bzw. $A_{out}(x)$ zur Teilchenmasse m übergehen.

Diese drei Forderungen an den Feldoperator $A(x)$ werden die einzige Voraussetzung unserer Theorie quantisierter, skalarer und lokaler Felder ohne gebundene Zustände bilden. Sie sollen, ihrer Bedeutung wegen, noch kurz erläutert werden.

Das Invarianzprinzip ist wohl so selbstverständlich, daß sich eine weitere Erörterung erübrigt. Hinsichtlich seiner mathematischen Formulierung sei auf die einschlägigen Arbeiten [1] verwiesen.

Die vorliegende Form der Kausalitätsbedingung [2] besagt ganz klar, daß Feldwirkungen in zueinander raumartig liegenden Punkten unabhängig sind. Sie dürfte anderen gebräuchlichen Kausalitätsbegriffen [3] äquivalent sein.

Über den anschaulichen Sinn der Asymptotenbedingung [4] ist zu sagen: Sie bringt zum Ausdruck, daß in jedem System wechselwirkender Bosonen, wenn man nur lange genug wartet, die Teilchen mit wachsender Zeit auseinanderstreben, um sich für genügend große Zeiten wie wechselwirkungsfreie Bosonen einer bestimmten Masse m zu verhalten. Die Asymptotenbedingung beschränkt demnach die Theorie auf solche Felder, die keine gebundenen Zustände zulassen, sondern nur Streuung, Erzeugung oder Vernichtung von Bosonen der Masse m beschreiben. Da im nächsten Kapitel aus der Asymptotenbedingung sehr weitgehende Folgerungen gezogen werden, ist eine präzise mathematische Fassung unerläßlich. Wir beginnen mit der Definition der ein- und auslaufenden Felder. $A_{in}(x)$ und $A_{out}(x)$ sollen natürlich den üblichen wechselwirkungsfreien Feldgleichungen und Vertauschungsrelationen genügen:

$$(\Box - m^2)A_{in}(x) = 0\,, \qquad [A_{in}(x), A_{in}(x')] = i\Delta(x - x')\,. \text{ [5]} \tag{2}$$

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[1] E. WIGNER: *Ann. of. Math.*, **30**, 149 (1939).

[2] Diese Form ist z.B. auch in der Arbeit M. GELL-MANN, M. L. GOLDBERGER und W. E. THIRRING: *Phys. Rev.*, **95**, 1612 (1954) gebraucht worden.

[3] E. C. G. STÜCKELBERG und G. WANDERS: *Acausalité de l'intéraction non-locale*, Manuskript, 1954.

[4] Über die prinzipielle Bedeutung einer Asymptotenbedingung für die Grundlagen der Feldtheorie vgl. R. HAAG: *On Quantum Field Theories*, Manuskript, 1954.

[5] Es wird außerdem verlangt, daß $A_{in}(x)$, $A_{out}(x)$ eine irreduzible Darstellung dieser Vertauschungsrelationen geben.

Es soll ferner Zustände Ω_{in} und Ω_{out} mit der Eigenschaft

$$A^{+}_{in}(x)\Omega_{in} = 0\,, \qquad A^{+}_{out}(x)\Omega_{out} = 0 \quad (6) \tag{3}$$

geben. Besondere Aufmerksamkeit erfordert die Präzision des Grenzübergangs $x_0 \to \pm\infty$. Es liegt vielleicht nahe

$$\lim_{x_0 \to -\infty} (\Phi, A(x)\Psi) = \lim_{x_0 \to -\infty} (\Phi, A_{in}(x)\Psi)$$

zu schreiben, doch sagt dies zu wenig aus, da die Matrixelemente der rechten Seite für normierbare Zustände punktweise gegen Null gehen. Wir führen stattdessen Operatoren

$$A^{f}(t) = i\int\limits_{x_0 = t}\left\{ A(x)\,\frac{\partial f(x)}{\partial x_0} - f(x)\,\frac{\partial A(x)}{\partial x_0}\right\} \mathrm{d}_3 x\,, \tag{4}$$

und entsprechend $A^{f}_{in}(t)$, $A^{f}_{out}(t)$ ein. Hier ist $f(x)$ eine beliebige normierte Lösung positiver Frequenzen der Klein-Gordongleichung, d.h.

$$(\Box - m^2)f(x) = 0\,, \qquad -i\int\left\{ f\,\frac{df^*}{\partial x_0} - f^*\,\frac{\partial f}{\partial x_0}\right\} \mathrm{d}_3 x = 1\,. \quad (7)$$

Der Asymptotenbedingung geben wir nun die Form

$$\begin{cases} \lim\limits_{\tau \to -\infty} (\Phi, A^{f}(\tau)\Psi) = (\Phi, A^{f}_{in}(t)\Psi) \\ \lim\limits_{\tau \to +\infty} (\Phi, A^{f}(\tau)\Psi) = (\Phi, A^{f}_{out}(t)\Psi) \end{cases} \tag{5}$$

worin die rechten Seiten unabhängig von t, also konstant sind.

Es folgen zum Abschluß des Kapitels noch einige leicht nachzuweisende Konsequenzen aus den genannten drei Forderungen.

Aus Ω_{in} läßt sich durch fortgesetzte Anwendung des Operators $A_{in}(x)$ ein vollständiges Orthonormalsystem des Hilbertraumes aufbauen, desgleichen ein zweites durch Anwendung von $A_{out}(x)$ auf Ω_{out}. Man braucht dazu lediglich ein System $f_\alpha(x)$ von Lösungen der Klein-Gordongleichung die nur positive

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(6) Diese Bedingungen müssen zur Kennzeichnung der freien Felder hinzugefügt werden, da die kanonischen Vertauschungsrelationen inäquivalente Darstellungen zulassen. Vgl. K. O. Friedrichs: *Math. Aspects of the Quantum Theorie of Fields* (New York, 1953).

(7) Der zeitunabhängige Operator $A^{f}_{in}(t)$ erzeugt also ein Teilchen mit der Wellenfunktion $f(x)$. Im Gegensatz zu $A_{in}(x)$ führen diese Operatoren bei Anwendung auf einen normierten Zustand nicht aus dem Hilbertraum heraus.

Frequenzen enthalten und der Orthonormalitätsrelation

$$-i\int\left\{f_\alpha\frac{\partial f_\beta^*}{\partial x_0}-f_\beta^*\frac{\partial f_\alpha}{\partial x_0}\right\}\mathrm{d}_3x=\delta_{\alpha\beta}\,,$$

sowie der Vollständigkeitsrelation

$$\sum_{\alpha=1}^{\infty}f_\alpha(x)f_\alpha^*(x')=i\Delta^+(x-x')$$

gehorchen. Die beiden Orthonormalsysteme sind dann:

$$\text{(6)}\qquad\begin{cases}\Omega_{\text{in}}\\ \Phi_{\text{in}}^{\alpha}=A_{\text{in}}^{\alpha}\Omega_{\text{in}}\\ \cdots\cdots\cdots\\ \Phi_{\text{in}}^{\alpha_1\ldots\alpha_k}=\dfrac{1}{\sqrt{p_{\alpha_1\ldots\alpha_k}}}A_{\text{in}}^{\alpha_1}\ldots A_{\text{in}}^{\alpha_k}\Omega_{\text{in}}\,,\end{cases}$$

$$\text{(7)}\qquad\begin{cases}\Omega_{\text{out}}\\ \Phi_{\text{out}}^{\alpha}=A_{\text{out}}^{\alpha}\Omega_{\text{out}}\\ \cdots\cdots\cdots\\ \Phi_{\text{out}}^{\alpha_1\ldots\alpha_k}=\dfrac{1}{\sqrt{p_{\alpha_1\ldots\alpha_k}}}A_{\text{out}}^{\alpha_1}\ldots A_{\text{out}}^{\alpha_k}\Omega_{\text{out}}.\end{cases}$$

Hierin bedeuten die Abkürzungen

$$A_{\text{out}}^{\alpha}=A_{\text{out}}^{f_\alpha}(t)\,,\qquad A_{\text{in}}^{\alpha}=A_{\text{in}}^{f_\alpha}(t)\,,\qquad p_{\alpha_1\ldots\alpha_k}=n_1!\,n_2!\ldots n_r!\,,$$

wenn je n_i der Indizes α_j untereinander gleich sind.

Die S-Matrix wird durch

$$\text{(8)}\qquad S_{(\alpha)(\beta)}=(\Phi_{\text{out}}^{(\alpha)},\,\Phi_{\text{in}}^{(\beta)})$$

definiert, wenn (α), (β) die Indexkombinationen

$$(\alpha)=\alpha_1,\ldots\alpha_k\,,\qquad(\beta)=\beta_1,\ldots,\beta_l$$

bedeuten. Es folgt unmittelbar die Unitarität der S-Matrix, sowie die Beziehung

$$\text{(9)}\qquad A_{\text{out}}=S^{\dagger}A_{\text{in}}S\,.$$

Gelegentlich sprechen wir von «gemischten Matrixelementen» des Feldoperators oder dessen Produktbildungen. Gemeint sind dann Ausdrücke der Form

$$(\Phi_{\text{out}}^{(\alpha)},\ A(x)\Phi_{\text{in}}^{(\beta)}),$$

die sich links auf einen Zustand des Orthonormalsystems (7), rechts auf einen Zustand des Systems (6) beziehen.

Aus Asymptotenbedingung und Invarianzprinzip folgt die Existenz eines Energie-Impulsvektors P_μ mit den Eigenschaften

$$-i\,[P_\mu,\ A(x)] = \frac{\partial A(x)}{\partial x_\mu}\,, \tag{10}$$

$$-i\,[P_\mu,\ A_{\text{in}}(x)] = \frac{\partial A_{\text{in}}(x)}{\partial x_\mu}\,, \qquad -i\,[P_\mu,\ A_{\text{out}}(x)] = \frac{\partial A_{\text{out}}(x)}{\partial x_\mu}\,,$$

ferner

$$\Omega_{\text{in}} = \Omega_{\text{out}} = \Omega \tag{11}$$

(der willkürliche Phasenfaktor ist gleich Eins gesetzt). P_μ erfüllt aus Invarianzgründen

$$P_\mu \Omega = 0.$$

Die Definition von $\Phi_{\text{in}}^{\alpha}$ bzw. $\Phi_{\text{out}}^{\alpha}$ zeigt, daß $(\Omega,\ A(x)\Phi_{\substack{\text{in}\\\text{out}}}^{\alpha})$ eine Lösung der Klein-Gordongleichung ist. Mit der Asymptotenbedingung folgt dann

$$\begin{cases} (\Omega,\ A(x)\Phi_{\text{in}}^{\alpha}) = (\Omega,\ A_{\text{in}}(x)\Phi_{\text{in}}^{\alpha}) = f_\alpha(x)\ (^8) \\ (\Omega,\ A(x)\Phi_{\text{out}}^{\alpha}) = (\Omega,\ A_{\text{out}}(x)\Phi_{\text{out}}^{\alpha}) = f_\alpha(x) \end{cases} \tag{12}$$

und daraus $\Phi_{\text{in}}^{\alpha} = \Phi_{\text{out}}^{\alpha}$. Eine weitere Folge der Asymptotenbedingung ist das Verschwinden des Vakuumerwartungswerts von $A(x)$:

$$(\Omega,\ A(x)\Omega) = (\Omega,\ A_{\text{in}}(x)\Omega) = 0\,. \tag{13}$$

3. – Die Reduktionsformel.

Nach vollzogener Grundlegung der Theorie wird man fragen, welche Feldtypen die genannten Voraussetzungen erfüllen und wie sich gegebenenfalls

(8) Diese Beziehungen, die sich hier als natürliche Folge der Asymptotenbedingung ergeben, sind früher von KÄLLÉN zur Festlegung der Renormierungskonstanten benutzt worden. G. KÄLLÉN: *Helv. Phys. Acta*, **25**, 417 (1952).

S-Matrix und Feldoperator berechnen lassen. Die allgemeine Operatorform der Voraussetzungen scheint zur Beantwortung dieser Fragen wenig geeignet. Es soll deshalb Aufgabe dieses Kapitels sein, ein den drei Forderungen äquivalentes Gleichungssystem zu entwickeln, das einer analytischen Behandlung zugänglich ist. Wir werden dabei den Umstand ausnützen, daß sich als Folge der Asymptotenbedingung die Matrixelemente von Feldoperator und S-Matrix geschlossen auf Vakuum-τ-Funktionen (damit sind Vakuumerwartungswerte von T-Produkten aus Feldoperatoren gemeint) zurückführen lassen. D.h.: Es genügt, sich auf die Vakuum-τ-Funktionen zu beschränken. Diese lassen sich aber im Prinzip aus einem unendlichen Gleichungssystem (System A) berechnen, das wir gegen Ende dieses Kapitels aufstellen werden. Die Ableitung des Systems A setzt allein die Asymptotenbedingung voraus. Invarianzprinzip und Kausalitätsforderung bewirken die Invarianz der Vakuum-τ-Funktionen. Somit wird die Aufgabe, alle Felder $A(x)$, die den drei Grundforderungen Invarianz, Kausalität und Asymptotenbedingung genügen, festzustellen und zu berechnen, auf die Diskussion der invarianten Lösungen des Systems A zurückgeführt (9).

Diese « Reduktion » von S-Matrix und Feldoperator ist nur ein Spezialfall der allgemeinen *Reduktionsformel*, die besagt, daß sich die gemischten Matrixelemente eines beliebigen T-Produktes von Feldoperatoren in elementarer Weise geschlossen durch die Vakuumerwartungswerte der T-Produkte ausdrücken lassen. Die Reduktionsformel ist eine Folge der Asymptotenbedingung. Sie wird im Anhang auf funktionale Weise in voller Allgemeinheit formuliert und bewiesen. Hier beweisen wir sie nur soweit, als zur Reduktion der S-Matrix und zur Aufstellung des Systems A notwendig ist.

Die Asymptotenbedingung sei in der Form (5) vorausgesetzt. Ferner gelte die Beziehung

$$\Omega_{\text{in}} = \Omega_{\text{out}} = \Omega \,.$$

Wir beweisen zuerst den Spezialfall

$$(\Omega,\, T(x_1, \ldots, x_n)\Phi^{\alpha}_{\text{in}}) = -i\int K_y \tau(x_1, \ldots, x_n, y) f_\alpha(y)\, \mathrm{d}^4 y \tag{14}$$

der Reduktionsformel, worin zur Abkürzung

$$T(x_1, \ldots, x_n) = T\{A(x_1) \ldots A(x_n)\}$$

$$K_y = \Box_y - m^2$$

(9) Man kann auch ausgehend von der Kommutatorbedingung (1) zu einer Analyse möglicher Feldtheorien kommen, vgl. dazu die in (4) zitierte Arbeit.

gesetzt ist, und

$$\tau(x_1, ..., x_n) = (\Omega, T(x_1, ..., x_n)\Omega) \tag{15}$$

die Vakuum-τ-Funktion von n Argumenten bedeutet. Aus der Definition von $\Phi^{\alpha}_{\text{in}}$ folgt:

$$(\Omega, T(x_1, ..., x_n)\Phi^{\alpha}_{\text{in}}) = (\bar{T}(x_1, ..., x_n)\Omega, A^{\alpha}_{\text{in}}\Omega) .$$

Anwendung der Asymptotenbedingung auf die rechte Seite gibt:

$$\begin{aligned}(\Omega, T(x_1, ..., x_n)\Phi^{\alpha}_{\text{in}}) &= \lim_{y_0 \to -\infty} (\bar{T}(x_1, ..., x_n)\Omega, A^{\alpha}(y_0)\Omega) \\ &= i \lim_{y_0 \to -\infty} \int (\Omega, T(x_1, ..., x_n, y) \overleftrightarrow{\frac{\partial}{\partial y_0}} f_{\alpha}(y)\, \mathrm{d}^3 y \\ &= -i \int \frac{\partial}{\partial y_0} \left\{ \tau_0(x_1, ..., x_n, y) \overleftrightarrow{\frac{\partial}{\partial y_0}} f_{\alpha}(y) \mathrm{d}^4 y \right\},\end{aligned}$$

mit den Abkürzungen:

$$A^{\alpha}(t) = A^{f_{\alpha}}(t) , \qquad f(x) \overleftrightarrow{\frac{\partial}{\partial x}} g(x) = f(x) \frac{\partial g(x)}{\partial x_0} - g(x) \frac{\partial f(x)}{\partial x_0} ,$$

da der Randterm bei $y_0 = +\infty$

$$i \lim_{y_0 \to +\infty} \int (\Omega, T(x_1, ..., x_n, y)\Omega) \overleftrightarrow{\frac{\partial}{\partial y_0}} f_{\alpha}(y)\, \mathrm{d}^3 y = (\Omega, A^{\alpha}_{\text{out}} T(x_1, ..., x_n)\Omega) ,$$

wegen

$$(A^{\alpha}_{\text{out}})^* \Omega = 0 ,$$

verschwindet.

$$i \int \frac{\partial}{\partial y_0} \left\{ \tau_0(x_1, ..., x_n, y) \overleftrightarrow{\frac{\partial}{\partial y_0}} f_{\alpha}(y) \right\} \mathrm{d}^4 y ,$$

gibt nach einigen Umformungen, die vom Verschwinden räumlicher Randterme Gebrauch machen, die rechte Seite der zu beweisenden Gleichung (14).

Nach diesem Muster beweist man ebenso die Beziehung

$$(\Omega, T(x_1, ..., x_n)\Phi^{\alpha_1 ... \alpha_{k+1}}_{\text{in}}) = \int K_y (\Omega, T(x_1, ..., x_n, y)\Phi^{\alpha_1 ... \alpha_k}_{\text{in}}) f_{\alpha_{k+1}}(y)\, \mathrm{d}^4 y . \tag{16}$$

Daraus folgt ohne weiteres durch vollständige Induktion die Formel

$$\begin{aligned}(\Omega, T(x_1 ... x_n)\Phi^{\alpha_1 ... \alpha_k}_{\text{in}}) &= \\ &= (-i)^k \int K_{y_1} ... K_{y_k} \tau(x_1 ... x_n y_1 ... y_k) f_{\alpha_1}(y_1) ... f_{\alpha_k}(y_k) \mathrm{d}^4 y_1 ... \mathrm{d}^4 y_k .\end{aligned} \tag{17}$$

Dies ist die Reduktionsformel für solche Matrixelemente von T-Produkten, die links auf das Vakuum bezogen sind. Danach können die zwischen Vakuum und einem beliebigen Zustand des Orthonormalsystems (6) genommenen Matrixelemente von T-Produkten aus den Vakuum-τ-Funktionen errechnet werden. Durch Übergang zu konjugiert komplexen Größen gewinnt man die Reduktionsformel für Matrixelemente von T-Produkten, die rechts auf das Vakuum bezogen sind.

Wir wenden uns nun der Reduktion der S-Matrix zu. Unsere Absicht ist, das Matrixelement

$$(\Phi_{\text{out}}^{(\alpha)}, \Phi_{\text{in}}^{(\beta)}) \qquad (\alpha) = \alpha_1, \ldots, \alpha_k, \quad (\beta) = \beta_1, \ldots, \beta_l$$

auf Vakuum-τ-Funktionen zurückzuführen, zunächst für den Fall, daß keiner der Indizes α_i mit einem der Indizes β_i übereinstimmt. Die Verallgemeinerungen von Gl. (16) lauten dann:

$$(\Phi_{\text{out}}^{(\alpha)}, T(x_1 \ldots x_n)\Phi_{\text{in}}^{(\beta)}) = -i\int K_\xi(\Phi_{\text{out}}^{(\alpha)}, T(x_1 \ldots x_n\xi)\Phi_{\text{in}}^{\beta_1 \ldots \beta_{l-1}})f_{\beta_l}(\xi)\,\mathrm{d}^4\xi\,,$$

$$(\Phi_{\text{out}}^{(\alpha)}, T(x_1 \ldots x_n)\Phi_{\text{in}}^{(\beta)}) = -i\int K_\eta(\Phi_{\text{out}}^{\alpha_1 \ldots \alpha_{k-1}}, T(x_1 \ldots x_n\eta)\Phi_{\text{in}}^{(\beta)})f_{\alpha_k}^*(\eta)\,\mathrm{d}^4\eta.$$

Durch vollständige Induktion folgt:

$$(\Phi_{\text{out}}^{(\alpha)}, \Phi_{\text{in}}^{(\beta)}) = (-i)^{k+l}\int K_{\xi_1} \ldots K_{\xi_k}K_{\eta_1} \ldots K_{\eta_l}\tau(\xi_1 \ldots \xi_k\eta_1 \ldots \eta_l)f^*(\xi_1) \ldots f^*(\xi_k)\cdot$$

$$\cdot f(\eta_1) \ldots f(\eta_l)\,\mathrm{d}^4\xi_1 \ldots \mathrm{d}^4\xi_k\,\mathrm{d}^4\eta_1 \ldots \mathrm{d}^4\eta_l\,.$$

Hieraus leitet man leicht eine Normalform für den durch

$$(\Phi_{\text{out}}^{(\alpha)}, \Phi_{\text{in}}^{(\beta)}) = (\Phi_{\text{in}}^{(\alpha)}, S\Phi_{\text{in}}^{(\beta)}) \tag{18}$$

definierten Operator S ab:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\int K_1 \ldots K_n\varphi(x_1 \ldots x_n) : A_{\text{in}}(x_1) \ldots A_{\text{in}}(x_n) : d^4x_1 \ldots d^4x_n\,. \tag{19}$$

Die Funktionen $\varphi(x_1, \ldots, x_n)$ bedeuten die Vakuum-φ-Funktionen [10], die aus den Vakuum-τ-Funktionen nach der Wick'schen Regel mit der Kontraktions-

[10] Zur Definition der τ- und φ-Funktionen vgl. z.B. die Arbeit W. ZIMMERMANN: *Suppl. al Nuovo Cimento*, **11**, 43 (1954), Abschnitt 4, deren Bezeichnungsweise wir hier übernehmen.

funktion Δ_F gebildet sind. Gl. (19) gilt ganz allgemein, unabhängig von der im Anfang gemachten Annahme über die Indizes. Es ist zu beachten, daß zur Ausführung der Integrale im Impulsraum die Vakuum-τ-Funktionen lediglich für solche Koordinaten bekannt sein brauchen, die auf dem Massenhyperboloid liegen. Nur die so eingeschränkten Vakuum-τ-Funktionen gehen in die Berechnung der S-Matrix ein.

Zum Abschluß leiten wir das unendliche Gleichungssystem A ab, das zur Bestimmung der Vakuum-τ-Funktionen dienen soll. Wir gehen von der einfachen Operatoridentität

$$T\{A(x_1) \dots A(x_n)\} = \sum_v \theta(x_1 - x_n) \dots \theta(x_{n-1} - x_n) T\{A(x_1) \dots A(x_{n-1})\} A(x_n)$$

aus, worin $\sum_v$ die Summation über alle $n-1$ Vertauschungen einer Koordinate x_i mit x_n bedeutet. Bildung des Vakuumerwartungswerts und Zerlegung der rechten Seite gibt:

$$(\Omega, T(x_1, \dots, x_n)\Omega) = \sum_v \sum_{(\alpha)} \theta(x_i - x_n) \dots \theta(x_{n-1} - x_n) (\Omega, T(x_1 \dots x_n) \Phi_{\text{in}}^{(\alpha)}) (\Phi_{\text{in}}^{(\alpha)}, A(x_n)\Omega) =$$

$$= \sum_v \sum_{k=1}^{\infty} \sum_{\alpha_1 \dots \alpha_k} \theta(x_1 - x_n) \dots \theta(x_{n-1} - x_n) (\Omega, T(x_1 \dots x_{n-1}) \Phi_{\text{in}}^{\alpha_1 \dots \alpha_k}) (\Phi_{\text{in}}^{\alpha_1 \dots \alpha_k}, A(x_n)\Omega) .$$

Drückt man jetzt noch auf der rechten Seite die Matrixelemente nach Reduktionsformel (17) durch Vakuum-τ-Funktionen aus, so hat man das gewünschte Gleichungssystem für die Vakuum-τ-Funktionen allein [11]:

$$\tau(x_1, \dots, x_n) = \sum_v \sum_{k=1}^{\infty} \theta(x_1 - x_n) \dots \theta(x_{n-1} - x_n) i^k \int d^4\xi_1 \dots d^4\xi_k \, d^4\eta_1 \dots d^4\eta_k$$

$$\text{(A)} \quad K_{\xi_1} \dots K_{\xi_k} \tau(x_1 \dots x_{n-1} \xi_1 \dots \xi_r) \Delta^+(\xi_1 - \eta_1) \dots \Delta^+(\xi_k - \eta_k) K_{\eta_1} \dots K_{\eta_k} \tau^*(x_n \eta_1 \dots \eta_k).$$

Aus Invarianzprinzip und Kausalitätsforderung folgt, daß die Vakuum-τ-Funktionen invariante Funktionen sind. Um eine Übersicht über alle den drei Forderungen Invarianz, Kausalität und Asymptotenbedingung gehorchenden Felder zu gewinnen, genügt es also, die invarianten Lösungen des Systems A zu untersuchen.

Wir erwähnen noch ein ganz ähnlich gebautes Gleichungssystem A′ für

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[11] In der üblichen Formulierung der Feldtheorie erfüllen die renormierten störungstheoretischen Entwicklungen der Vakuum-τ-Funktionen das System A identisch. Der Beweis kann analog den in der Notiz W. Zimmermann: *Nuovo Cimento*, **11**, 416 (1954) skizzierten Überlegungen durchgeführt werden.

die Funktionen τ, das sich ausgehend von der Operatorenidentität

$$T\{A(x_1)\dots A(x_n)\} = \sum_v \theta(x_n - x_1)\dots\theta x_n - x_{n-1})A(x_n)T\{A(x_1)\dots A(x_{n-1})\}$$

ebenso erhalten läßt. Das System A' ist dem System A äquivalent

Ein weiteres Gleichungssystem der Vakuum-τ-Funktionen, das für praktische Anwendungen geeignet erscheint, kann aus der Identität

$$\sum_{(\beta)} (\Phi_{\text{in}}^{(\alpha)}, \Phi_{\text{out}}^{(\beta)})(\Phi_{\text{out}}^{(\beta)}, TA(x_1)A(x_2)\Phi_{\text{in}}^{(\gamma)}) = \tag{20}$$

$$= \theta(x_1 - x_2) \sum_{(\beta)} (\Phi_{\text{in}}^{(\alpha)}A(x_1)\Phi_{\text{out}}^{(\beta)})(\Phi_{\text{out}}^{(\beta)}, A(x_2)\Phi_{\text{in}}^{(\gamma)}) + \text{Symm.}$$

durch Einsetzen der Reduktionsformel gewonnen werden. Auch dieses System ist dem System A äquivalent.

4. – Diskussion des Gleichungssystems A.

Angesichts der wenigen Voraussetzungen, die zur Ableitung des Systems A erforderlich waren, wird man erwarten, daß diese Gleichungen eine große Zahl von Lösungen besitzen. Dieser Abschnitt verfolgt das Ziel, einen Überblick über die invarianten Lösungen zu geben und insbesondere zu zeigen, wie man durch Stellung von Randbedingungen bestimmte Lösungen aussondern kann. Wir beschränken uns dabei auf eine störungstheoretische Diskussion der Gleichungen.

Zunächst sei vermerkt, daß selbstverständlich die bekannten τ-Funktionen eines freien Feldes eine exakte Lösung des Systems A bilden. Sie lauten:

$$\tau(x_1, x_2) = \Delta_F(x_1 - x_2)\,; \qquad \tau(x_1, \dots, x_{2n+1}) = 0\,,$$

$$\tau(x_1, \dots, x_{2n}) = \frac{1}{n} \sum_{\mu<\nu} \Delta_F(x_\mu - x_\nu)\tau(x_1 \dots x_{\mu-1}, x_{\mu+1} \dots x_{\nu-1}, x_{\nu+1} \dots x_{2n})\,. \tag{21}$$

Für die weitere Behandlung ist es zweckmässig, zu den in Kap. 3 bereits eingeführten Funktionen $\varphi(x_1 \dots x_n)$ überzugehen. Man kann das System leicht in ein Gleichungssystem für die φ-Funktionen umformen (vgl. Anhang (A.11)). Wir wollen nun die φ-Funktionen in einer ersten vom freien Feld abweichenden Näherung bestimmen. Es wird sich zeigen, daß diese Berechnung eine über die 1. Näherung hinausgehende Bedeutung hat und eine Übersicht über die möglichen Lösungen des Systems A in beliebiger Näherung vermittelt. Es sei

$$\varphi(x_1, \dots, x_n) = g\varphi^{(1)} + g^2\varphi^{(2)} + \dots\,.$$

Wir nehmen an, daß in 1. Näherung ($\sim g$) nur eine bestimmte Funktion

$\varphi(x_1 \dots x_n)$ von Null verschieden ist, d.h. vom freien Feld abweicht [12]. Aus (A) folgt dann für $\varphi^{(1)}(x_1, \dots, x_n)$

$$(22) \qquad \varphi^{(1)}(x_1, \dots x_n) =$$

$$= \sum_v \theta(x_1 - x_n) \dots \theta(x_{n-1} - x_n) \left\{ \int K_{\xi_n} \varphi^{(1)}(x_1, \dots, x_{n-1}, \xi_n) \Delta^+(\xi_n - x_n) \, \mathrm{d}^4\xi_n + \right.$$

$$\left. + i^{2(n-1)} \int \Delta^+(x_1 - \xi_1) \dots \Delta^+(x_{n-1} - \xi_{n-1}) K_{\xi_1} \dots K_{\xi_{n-1}} \varphi^{(1)*}(x_n, \xi_1, \dots, \xi_{n-1}) \, \mathrm{d}^4\xi_1 \dots \mathrm{d}^4\xi_{n-1} \right\}.$$

Es ist bequem, diese Gleichung nicht nach $\varphi(x_1 \dots x_n)$, sondern nach einer Funktion $\omega(x_1 \dots x_n)$ aufzulösen, mit

$$(23) \qquad \varphi(x_1 \dots x_n) = \int \Delta_F(x_1 - \xi_1) \dots \Delta_F(x_n - \xi_n) \omega_n(\xi_1, \dots, \xi_n) \, \mathrm{d}^4\xi_1 \dots \mathrm{d}^4\xi_n \, .$$

(22) geht damit über in

$$(22') \qquad \int \Delta_F(x_1 - \xi_1) \dots \Delta_F(x_n - \xi_n) \omega^{(1)}(\xi_1, \dots, \xi_n) \, \mathrm{d}^4\xi_1 \dots \mathrm{d}^4\xi_n =$$

$$= \sum_v \theta(x_1 - x_n) \dots \theta(x_{n-1} - x_n) \cdot$$

$$\cdot \int \{ \Delta_F(x_1 - \xi_1) \dots \Delta_F(x_{n-1} - \xi_{n-1}) (-i) \Delta^-(x_n - \xi_n) \omega^{(1)}(\xi_1, \dots, \xi_n) +$$

$$+ i\Delta^+(x_1 - \xi_1) \dots i\Delta^+(x_{n-1} - \xi_{n-1}) \Delta_F(x_n - \xi_n) \omega^{(1)*}(\xi_1 \dots \xi_n) \} \, \mathrm{d}^4\xi_1 \dots \mathrm{d}^4\xi_n \, .$$

Mit Anwendung der Beziehungen

$$-i\Delta^-(x) = \Delta_F(x) + i\Delta_R(x) \, ; \qquad i\Delta^+(x) = \bar{\Delta}_F(x) - i\Delta_R(x)$$

folgt aus (22')

$$(24) \qquad \sum_v \theta(x_1 - x_n) \dots \theta(x_{n-1} - x_n) \cdot$$

$$\cdot \int \{ \Delta_F(x_1 - \xi_1) \dots \Delta_F(x_{n-1} - \xi_{n-1}) \Delta_R(x_n - \xi_n) \omega^{(1)}(\xi_1, \dots \xi_n) +$$

$$+ i\Delta^+(x_1 - \xi_1) \dots i\Delta^+(x_{n-1} - \xi_{n-1}) \Delta_R(x_n - \xi_n) \omega^{(1)*}(\xi_1, \dots, \xi_n) \} \, \mathrm{d}^4\xi_1 \dots \mathrm{d}^4\xi_n = 0 \, .$$

Es wird sich herausstellen, daß invariante Funktionen $\omega^{(1)}$ nur dann ungleich Null sind, wenn alle Zeiten $x_{10}, \dots, x_{n0}$ übereinstimmen. Wegen der Invarianz

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[12] Dies entspricht dem Fall, daß bei einer Lagrangeformulierung ein Kopplungsterm mit n Feldoperatoren vorhanden ist.

folgt daraus, daß auch die räumlichen Komponenten der Argumente gleich sein müssen; d.h. die Funktionen $\omega^{(1)}$ bestehen aus Produkten von vierdimensionalen δ-Funktionen und deren Ableitungen.

Wir beweisen diese Aussagen zunächst für $\omega^{(1)}(x_1, x_2, x_3)$ und übertragen dann das Resultat auf beliebiges n. Für $\omega^{(1)}(x_1, x_2, x_3)$ lautet (24)

$$(24') \quad \theta(x_1 - x_3)\theta(x_2 - x_3)\int\{\Delta_F(x_1 - \xi_1)\Delta_F(x_2 - \xi_2)\Delta_R(x_3 - \xi_3)\omega^{(1)}(\xi_1, \xi_2, \xi_3) - \\ - \Delta^+(x_1 - \xi_1)\Delta^+(x_2 - \xi_2)\Delta_R(x_3 - \xi_3)\omega^{(1)*}(\xi_1\xi_2\xi_3)\}\, \mathrm{d}^4\xi_1\, \mathrm{d}^4\xi_2\, \mathrm{d}^4\xi_3 + \\ + \theta(x_2 - x_1)\theta(x_3 - x_1)\{\} + \theta(3, 1; 2)\{\} = 0 .$$

Durch Anwendung von Klein-Gordonoperatoren folgt

$$\omega^{(1)}(x, x_2, x_3) = 0 \quad \text{wenn} \begin{cases} x_{10},\, x_{20} > x_{30} \\ x_{20},\, x_{30} > x_{10} \\ x_{30},\, x_{10} > x_{20} . \end{cases}$$

Aus dem in Kap. 3 erwähnten Gleichungssystem (A′) folgt ebenso, daß

$$\omega^{(1)}(x_1, x_2, x_3) = 0 \quad \text{wenn} \begin{cases} x_{10},\, x_{20} < x_{30} \\ x_{20},\, x_{30} < x_{10} \\ x_{30},\, x_{10} < x_{20} . \end{cases}$$

Diese Bedingungen besagen, daß in $\omega^{(1)}(x_1, x_2, x_3)$ alle Zeiten gleich sein müssen. Unter den danach verbleibenden invarianten Funktionen gibt es drei Lösungen der Gleichung (24′). Sie lauten im Orts- bzw. im Impulsraum (g_{31}, g_{32}, g_{33} sind reelle Konstanten):

1) $\omega^{(1)} = ig_{31}\,\delta(x_1 - x_2)\,\delta(x_2 - x_3)$; bzw. $ig_{31}\cdot\delta(p_1 + p_2 + p_3)$,

2) $\omega^{(1)} = ig_{32}(\Box_{x_1} + \Box_{x_2} + \Box_{x_3})\,\delta(x_1 - x_2)\delta(x_2 - x_3)$;
bzw. $-ig_{32}(p_1^2 + p_2^2 + p_3^2)\cdot\delta(p_1 + p_2 + p_3)$,

3) $\omega^{(1)} = ig_{33}(\Box_1\partial_{\mu_2}\partial_{\mu_3} + \Box_2\partial_{\mu_3}\partial_{\mu_1} + \Box_3\partial_{\mu_1}\partial_{\mu_2})\,\delta(x_1 - x_2)\,\delta(x_2 - x_3)$;
bzw. $ig_{33}(p_1^2(p_2p_3) + p_2^2(p_3p_1) + p_3^2(p_1p_2))\delta(p_1 + p_2 + p_3)$.

Wir verifizieren dies am Beispiel der Lösung (1). Hierfür lautet (24′)

$$\theta(1, 2; 3)\int\{\Delta_F(x_1 - \xi)\Delta_F(x_2 - \xi) + \Delta^+(x_1 - \xi)\Delta^+(x_2 - \xi)\}\cdot \\ \cdot\Delta_R(x_3 - \xi)\,\mathrm{d}^4\xi + \theta(2, 3;\ 1) \ldots + \ldots = 0 .$$

Diese Gleichung ist erfüllt, da wegen $x_{10},\ x_{20} > x_{30} > \xi_0$ die Δ_F-Funktionen durch $i\Delta^+$ ersetzt werden können.

Es sind jedoch nicht beliebig hohe Ableitungen von δ-Funktionen Lösungen, da im allgemeinen durch die Differentiation der Δ_F-Funktionen Zusatzterme auftreten, die die Äquivalenz von Δ_F und $i\Delta^+$ zerstören. Man kann zeigen, daß es nur die angegebenen drei Lösungen gibt.

Für $\omega^{(1)}(x_1, ..., x_n)$ erhält man ähnliche Resultate. Man stellt wieder fest, daß Produkte von δ-Funktionen und einige Ableitungen dieser Funktionen die Gleichungen (24) erfüllen. Für $\omega^{(1)}(x_1, ..., x_n)$ gibt es n unabhängige Lösungen in denen willkürliche Parameter $g_{n1}, ..., g_{nn}$ auftreten. Im Impulsraum stellen sie symmetrische Polynome in den Variablen $p_1, ..., p_n$ dar, die dadurch charakterisiert sind, daß die Potenzsumme je zweier Impulse kleiner als vier ist.

Die so gewonnenen Funktionen $\varphi^{(1)}$ (bzw. $\omega^{(1)}$) stimmen mit den Ausdrücken überein, die man bei Benutzung der üblichen Formulierung von Feldtheorien in 1. Näherung erhält. Dabei entsprechen natürlich δ'-Funktionen Theorien mit Ableitungskopplung.

Es zeigt sich also, daß die hier diskutierten Gleichungen formal (d.h. bis auf Renormierungsterme) den konventionellen Feldgleichungen äquivalent sind. Wenn wir z.B. die Konstanten $g_{31}, g_{41}, ..., g_{n1}$ als von Null verschieden vorgeben, so entspricht dies einer Lagrangefunktion mit dem Kopplungsterm

$$L' = g'_{31}A^3(x) + g'_{41}A^4(x) + ... + g'_{n1}A^n(x) .$$

Die Konstanten g_{i1} und g'_{i1} stimmen dabei in erster Näherung überein.

Durch die angegebenen Lösungen der (homogenen) Gleichungen für die $\varphi^{(1)}$-Funktionen sind Parameter $g_{n1}, ..., g_{nn}$ $(n=3, 4, ...)$ eingeführt worden, die als Kopplungskonstanten fungieren. Wenn diese Konstanten vorgegeben sind (also $\varphi^{(1)}$ bekannt ist), so kann man versuchen, die φ-Funktionen höherer Näherung in einer Potenzreihenentwicklung nach diesen Konstanten mit Hilfe des Systems A zu bestimmen. In 2. Näherung erhält man für $\varphi^{(2)}$ eine inhomogene Gleichung, in deren inhomogenen Gliedern die Funktionen 1. Näherung auftreten. Der homogene Teil der Gleichung für $\varphi^{(2)}$ ist identisch mit der schon behandelten Gleichung für $\varphi^{(1)}$. $\varphi^{(2)}$ ist also bis auf Lösungen dieser homogenen Gleichung bestimmt. Dieser Sachverhalt gilt in beliebiger Ordnung der Störungsrechnung.

Die notwendige Festlegung der bei der Durchführung der Störungsrechnung auftretenden homogenen Lösungen kann nun dadurch geschehen, daß man an die Lösungen des Systems A Randbedingungen stellt, indem man einiges über das asymptotische Verhalten der Funktionen ω vorgibt. Im einparametrigen Fall (etwa nur $g_{n1} \neq 0$) lautet diese Vorgabe: $\omega(x_1, ..., x_n)$ soll (in jeder Näherung) für große Impulse einen konstanten Term g_{n1} enthalten; es soll keine Terme der Form $g_{n2}(p_1^2 + ... p_n^2)$; $g_{n3}(p_1^2 p_2 p_3 + ...)$, etc., enthalten, die den anderen Lösungen der homogenen Gleichung für $\omega^{(1)}(x_1 ... x_n)$ entsprächen. Alle anderen Funktionen ω sollen asymptotisch keine Terme enthalten, die

konstant sind oder den homogenen Gleichungen entsprechende Polynome darstellen. Allgemein gibt man also die Kopplungskonstanten und damit ein diesen Konstanten entsprechendes asymptotisches Verhalten der Funktionen ω vor ([13]).

Bekanntlich gibt es unter den einparametrigen Theorien genau zwei, die im Rahmen der üblichen Formulierung renormierbar sind. Für diese ist g_{31} bzw. g_{41} ungleich Null. (Kopplung $\sim A^3$ bzw. $\sim A^4$). Mit diesen Randbedingungen hat das System A in jeder Näherung eindeutig bestimmte Lösungen, die den renormierten Ausdrücken entsprechen ([14]).

Die in diesem Kapitel gemachten Aussagen beziehen sich natürlich nur auf die Störungsrechnung (beliebiger Ordnung). Insofern kommt hier einer der Vorzüge des Systems A, das eine Diskussion renormierbarer Theorien ohne Einschränkung auf die Störungsrechnung ermöglichen sollte, nicht zur Geltung. Jedoch dürfte auch die störungsmäßige Behandlung mit dem System A konsequenter als das sonst übliche Verfahren sein, da hier nur mit divergenzfreien Gleichungen gearbeitet wird. Auf die Einzelheiten der Störungsrechnung gehen wir hier nicht ein, da dies angesichts des Modellcharakters des zugrundegelegten skalaren Feldes nicht lohnend erscheint.

Anhang

Funktionale Formulierungen.

Der ein Funktional der Raumzeitfunktion $J(x)$ darstellende geordnete Operator ([15])

$$T \exp\left[i \int_{t_1}^{t_2} A(x) J(x)\, \mathrm{d}x\right] \equiv \mathfrak{C}\{t_2, t_1; J\}$$

ist durch die Integralgleichung

$$\mathfrak{C}\{t_2, t_1; J\} = 1 + i \int_{t_1}^{t_2} \mathfrak{C}\{t_2, x_0; J\} J(x) A(x)\, \mathrm{d}x \tag{A.1}$$

([13]) Diese Vorgabe entspricht im Ortsraum einer Festlegung der Funktionen ω für gleiche Argumente durch δ-Funktionen und deren Ableitungen.

([14]) Für die nichtrenormierbaren Theorien gibt es innerhalb der Störungsrechnung keine Lösungen des Systems A; z.B. existiert für derartige Randbedingungen die Δ'_F-Funktion in g^2-Näherung nicht.

([15]) Zu den Eigenschaften dieses von Schwinger eingeführten Operators vgl. K. Symanzik: *Zeits. f. Naturf.*, **10***a*, 809 (1954) und dort angegebene Literatur.

bestimmt. Zur Abkürzung setzen wir

$$\mathcal{C}\{+\infty, -\infty; J\} \equiv \mathcal{C}\{J\} \quad \text{und} \quad \langle|\mathcal{C}\{J\}|\rangle \equiv \mathcal{C}_0\{J\},$$

wobei wir der Übersichtlichkeit halber die Diracsche Schreibweise mit dem Vakuum $|\rangle$ verwenden.

Zufolge

$$\mathcal{C}_0\{J\} = \sum_{n=0}^{\infty} \frac{i^n \tau_n\{J\}}{n!},$$

mit

$$\tau_n\{J\} \equiv \int \dots \int \mathrm{d}x_1 \dots \mathrm{d}x_n \, \tau(x_1 \dots x_n) J(x_1) \dots J(x_n),$$

ist dieses Funktional das erzeugende Funktional der Vakuum-τ-Funktionen (Gl. (15)), die sich aus $\mathcal{C}_0\{J\}$ durch funktionales Differenzieren und Nullsetzen von J gewinnen lassen. Den Operator

$$\mathcal{C}\{J\} = \sum_{(\alpha)} \sum_{(\beta)} |(\alpha)\rangle\langle(\alpha)|\mathcal{C}\{J\}|(\beta)\rangle\langle(\beta)|$$

selbst gewinnen wir, indem wir für $|(\alpha)\rangle$ das Orthonormalsystem der auslaufenden asymptotisch freien Teilchen, für $|(\beta)\rangle$ das der einlaufenden Teilchen einsetzen (Gl. (7), (6)). Benutzung der Asymptotenbedingung (5) und der aus (A.1) folgenden Relation

$$\text{(A.2)} \qquad \frac{\delta}{\delta J(x)} \mathcal{C}\{J\} = i\mathcal{C}\{+\infty, x_0; J\} A(x) \mathcal{C}\{x_0, -\infty; J\}$$

gibt

$$\text{(A.3)} \qquad \langle(\alpha)|\mathcal{C}\{J\}|(\beta)\rangle = \langle(\alpha)|\mathcal{C}\{J\} \frac{1}{\sqrt{P_{(\beta)}}} \prod_{\nu=1}^{n} \left[i \int A_{\mathrm{in}}^{(-)}(x) \overleftrightarrow{\partial_0} f_{\beta\nu}(x) \, \mathrm{d}_3 x \right] |\rangle =$$

$$= \frac{1}{\sqrt{P_{(\beta)}}} \prod_{\nu=1}^{n} \left[\lim_{x_0 \to -\infty} \int \frac{\delta}{\delta J(x)} \overleftrightarrow{\partial_0} f_{\beta\nu}(x) \, \mathrm{d}_3 x \right] \langle(\alpha)|\mathcal{C}\{J\}|\rangle$$

mit der Abkürzung

$$f(x) \overleftrightarrow{\partial_0} g(x) \equiv -g(x) \frac{\partial}{\partial x_0} f(x) + f(x) \frac{\partial}{\partial x_0} g(x).$$

Entsprechend wird für $\varphi_{(\alpha)}$ eingesetzt. Mit der Vollständigkeitsrelation erhalten wird durch Summation über alle Anfangs- und Endzustände:

$$\text{(A.4)} \quad \mathcal{C}\{J\} = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{1}{n!\, n'!} \left[\sum_{\alpha} i \int A_{\mathrm{out}}^{(-)}(x) \overleftrightarrow{\partial_0} f_\alpha(x) \, \mathrm{d}_3 x \cdot \lim_{x_0 \to +\infty} \int f_\alpha^*(x') \overleftrightarrow{\partial_0'} \frac{\delta}{\delta J(x')} \, \mathrm{d}_3 x' \right]^n |\rangle \cdot$$

$$\cdot \langle| \left[\sum_{\beta} \lim_{y_0 \to -\infty} \int \frac{\delta}{\delta J(y)} \overleftrightarrow{\partial_0} f_\beta(y) \, \mathrm{d}_3 y \cdot i \int f_\beta^*(y') \overleftrightarrow{\partial_0'} A_{\mathrm{in}}^{(+)}(y') \, \mathrm{d}_3 y' \right]^{n'} \mathcal{C}_0\{J\} =$$

$$= \exp\left[\lim_{x_0 \to \infty} \int A_{\mathrm{out}}^{(-)}(x) \overleftrightarrow{\partial_0} \frac{\delta}{\delta J(x)} \, \mathrm{d}_3 x \right] |\rangle \langle| \exp\left[\lim_{y_0 \to -\infty} \int \frac{\delta}{\delta J(y)} \overleftrightarrow{\partial_0} A_{\mathrm{in}}^{(+)}(y) \, \mathrm{d}_3 y \right] \mathcal{C}_0\{J\}.$$

Um die Randintegrale durch Volumenintegrale ersetzen zu können, ist zu erreichen, daß die Integrale über den jeweils gegenüberliegenden Rand verschwinden. In (A.4) ist dies noch nicht der Fall. (Denn (A.3) folgt aus (A.2) und der Asymptotenbedingung nur dann, wenn $J(x)$ am Rande hinreichend stark verschwindet; dann aber dürfen in der Umgebung des Randes weitere funktionale Ableitungen nicht beliebig vorgenommen werden, was der Grund für das Nichtverschwinden der bei der Umformung von (A 4) auftretenden Randintegrale ist.) Daher gehen wir durch

$$\mathcal{C}\{J\} = \exp\left[-\frac{J\Delta_F J}{2}\right] \mathcal{S}_0\{J\}, \tag{A.5}$$

mit

$$\mathcal{S}_0\{J\} = \sum_{n=0}^{\infty} \frac{i^n \varphi_n\{J\}}{n!} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \ldots \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, \varphi(x_1 \ldots x_n) J(x_1) \ldots J(x_n)$$

zum erzeugenden Funktional der Vakuum-φ-Funktionen über [10]. Zur Abkürzung ist hier und in ähnlichen späteren Ausdrücken Matrixmultiplikation vorausgesetzt, also

$$J\Delta_F J \equiv \iint J(x)\Delta_F(x - x')J(x')\,\mathrm{d}x\,\mathrm{d}x'$$

$$A_{\mathrm{in}} \boldsymbol{K} \frac{\delta}{\delta J} \equiv \int A_{\mathrm{in}}(x)(\Box_x - m^2) \frac{\delta}{\delta J(x)}\,\mathrm{d}x \qquad \text{usw.}$$

Einsetzen von (A.5) in (A.4) gibt

$$\mathcal{C}\{J\} = \exp\left[\int\limits_{x_0=\infty} A^{(-)}_{\mathrm{out}}(x) \overleftrightarrow{\partial_0} \frac{\delta}{\delta J(x)}\,\mathrm{d}_3x\right] \exp\left[\frac{-J\Delta_F J}{2}\right] |\rangle \cdot$$

$$\cdot \langle | \exp\left[-\int\limits_{y_0=-\infty} J(y)\Delta_F(y - y') \overleftrightarrow{\partial'_0} A^{(+)}_{\mathrm{in}}(y')\,\mathrm{d}y\,\mathrm{d}_3y'\right] \cdot$$

$$\cdot \exp\left[\int\limits_{z_0=-\infty} \frac{\delta}{\delta J(z)} \overleftrightarrow{\partial_0} A^{(+)}_{\mathrm{in}}(z)\,\mathrm{d}_3z\right] \mathcal{S}_0\{J\} =$$

$$= \exp\left[-\frac{J\Delta_F J}{2}\right] \exp\left[\int\limits_{x_0=\infty} J(x)\Delta_F(x - x') \overleftrightarrow{\partial'_0} A^{(-)}_{\mathrm{out}}(x')\,\mathrm{d}_3x'\,\mathrm{d}x \cdot\right.$$

$$\cdot \exp\left[\int\limits_{y_0=\infty} A^{(-)}_{\mathrm{out}}(y) \overleftrightarrow{\partial_0} \frac{\delta}{\delta J(y)}\,\mathrm{d}_3y\right] S^+ \cdot \exp\left[-\int\limits_{z_0=-\infty} J'(z)\Delta_F(z - z') \overleftrightarrow{\partial'_0} A^{(+)}_{\mathrm{in}}\,\mathrm{d}z\,\mathrm{d}_3z'\right] \cdot$$

$$\cdot \exp\left[\int\limits_{z_0=-\infty} \frac{\delta}{\delta J(z'')} \overleftrightarrow{\partial''_0} A^{(+)}_{\mathrm{in}}(z'')\,\mathrm{d}_3z''\right] \varphi_0\{J\} \Big|_{J'=J}$$

oder

$$\text{(A.6a)} \qquad S\mathcal{C}\{J\} = \exp\left[-\frac{J\Delta_F J}{2}\right] : \exp[iJA_{\text{in}}] \exp\left[-A_{\text{in}}\boldsymbol{K}\frac{\delta}{\delta J}\right] : \mathcal{S}_0\{J\}$$

und

$$\text{(A.6b)} \qquad \mathcal{C}\{J\}S = \exp\left[-\frac{J\Delta_F J}{2}\right] : \exp[iJA_{\text{out}}] \exp\left[-A_{\text{out}}\boldsymbol{K}\frac{\delta}{\delta J}\right] : \mathcal{S}_0\{J\}$$

und mit $J = 0$ schließlich

$$\text{(A.7)} \qquad S = : \mathcal{S}_0\{-A_{\text{in}}\boldsymbol{K}\} : = : \mathcal{S}_0\{-A_{\text{out}}\boldsymbol{K}\} :$$

(Bei der Rechnung sind der Reihe nach die Volterra-Formel

$$\text{(A.8)} \qquad \exp\left[J'\frac{\delta}{\delta J}\right]\mathcal{F}\{J\} = \mathcal{F}\{J+J'\}\exp\left[J'\frac{\delta}{\delta J}\right]$$

($\mathcal{F}\{J\}$ ein beliebiges Funktional),

$$\int\limits_{x_0, x_0' = \mp\infty} A_{\substack{\text{in}\\\text{out}}}^{(\pm)}(x)\overleftrightarrow{\partial_0}\Delta_F(x-x')\overleftrightarrow{\partial_0'}A_{\substack{\text{in}\\\text{out}}}^{(\pm)}(x')\,\mathrm{d}_3x\,\mathrm{d}_3x' = 0\,,$$

$$S = : \exp\left[i\int A_{\text{in}}^{(-)}(x)\overleftrightarrow{\partial_0}A_{\text{out}}^{(+)}(x)\,\mathrm{d}_3x\right] :$$

und

$$A_{\text{out}}(x) = S^+A_{\text{in}}(x)S\,, \qquad A_{\text{in}}(x) = SA_{\text{out}}(x)S^+$$

verwendet worden. Dabei ist in der zuletzt benutzten Formel für die S-Matrix zwischen den $A_{\text{in}}^{(-)}$- und den $A_{\text{out}}^{(+)}$-Operatoren anstelle des Einheitsoperators der Operator $|\rangle\langle|$ eingesetzt zu denken. Die in (A.6) erfolgte Ersetzung der Randintegrale durch Volumenintegrale, welche bei (A.4) nicht zulässig war, ist hier erlaubt.)

(A.7) ist die Normalform der S-Matrix (Gl. (19)), (A.6a,b) sind die in Kap. 3 erwähnten allgemeinen Reduktionsfo meln, durch die Matrixelemente geordneter Operatoren auf Vakuumerwartungswerte zurückgeführt werden.

Mit

$$\widehat{J}(x', x) \equiv J(x')\theta(x'-x) \qquad \text{und} \qquad \overline{\mathcal{C}}\{J\} = (\mathcal{C}\{J\})^{\#}$$

läßt sich (A.1) auch schreiben:

$$\mathcal{C}\{J\} = 1 - \int\limits_{x_0=-\infty}^{+\infty} J(x)\mathcal{C}\{\widehat{J}\}\frac{\delta}{\delta J''(x)}\overline{\mathcal{C}}\{J''\}\,\mathrm{d}x\Big|_{J''=0}.$$

Bilden des Vakuumerwartungswerts und Benutzung von (A.4) gibt die funktio-

nale Form des Systems A:

$$(A.9)\quad \mathcal{C}_0\{J\} = 1 - \int_{x_0=-\infty}^{\infty} J(x) \langle | \exp\left[\int_{x_0=-\infty} \frac{\delta}{\delta J'(x')} \overleftrightarrow{\partial_0'} A_{\text{in}}^{(+)}(x') \mathrm{d}_3 x'\right] \cdot$$

$$\cdot \exp\left[-\int_{x_0''=-\infty} A_{\text{in}}^{(-)}(x'') \overleftrightarrow{\partial_0'} \frac{\delta}{\delta J''(x'')} \mathrm{d}_3 x''\right] | \rangle \cdot \frac{\delta}{\delta J''(x)} \mathcal{C}_0\{J'\} \overline{\mathcal{C}}_0\{J''\} \,\mathrm{d}x \Big|_{\substack{J'=\widehat{J} \\ J''=0}} =$$

$$= 1 - \int_{x_0=-\infty}^{\infty} J(x) \frac{\delta}{\delta J''(x)} \exp\left[-\int\int_{x_0', x_0''=-\infty} \frac{\delta}{\delta J'(x')} \overleftrightarrow{\partial_0'} i\Delta^{(+)}(x'-x'') \overleftrightarrow{\partial_0''} \frac{\delta}{\delta J''(x'')} \mathrm{d}_3 x' \,\mathrm{d}_3 x''\right] \cdot$$

$$\cdot \mathcal{C}_0\{J'\} \overline{\mathcal{C}}_0\{J''\} \,\mathrm{d}x \Big|_{\substack{J'=\widehat{J} \\ J''=0}} =$$

$$= 1 - \int_{x_0=-\infty}^{\infty} J(x) \frac{\delta}{\delta J''(x)} \exp\left[\frac{\delta}{\delta J'} \overleftarrow{K} i\Delta^{(+)} \overrightarrow{K} \frac{\delta}{\delta J''}\right] \mathcal{C}_0\{J'\} \overline{\mathcal{C}}_0\{J''\} \,\mathrm{d}x \Big|_{\substack{J'=\widehat{J} \\ J''=0}}.$$

Hieraus folgt wegen

$$(A.10)\quad \mathcal{C}_0\{J\} = 1 - \int_{-\infty}^{\infty} \mathrm{d}x_0 \frac{\mathrm{d}}{\mathrm{d}x_0} \mathcal{C}_0\{\widehat{J}\} = 1 + \int_{x_0=-\infty}^{\infty} J(x) \frac{\delta}{\delta J'(x)} \mathcal{C}_0\{J'\} \,\mathrm{d}x \Big|_{J'=\widehat{J}}$$

nach Einsetzen von (A.5) und mehrmaliger Benutzung von (A.8)

$$\int_{x_0=-\infty}^{\infty} J(x) \exp\left[-\frac{\widehat{J} \Delta_F \widehat{J}}{2}\right] \left(-\int \Delta_F(x-x') \widehat{J}(x', x) \,\mathrm{d}x' + \frac{\delta}{\delta J'(x)}\right) \mathcal{S}_0\{J'\} \,\mathrm{d}x \Big|_{J'=\widehat{J}} =$$

$$= -\int_{x_0=-\infty}^{\infty} J(x) \exp\left[-\frac{\widehat{J} \Delta_F \widehat{J}}{2}\right] \left[\int \Delta^{(-)}(x-x') \left(-i \widehat{J}(x', x) + \boldsymbol{K}_{x'} \frac{\delta}{\delta J'(x')}\right) \mathrm{d}x' + \frac{\delta}{\delta J''(x)}\right] \cdot$$

$$\cdot \exp\left[\frac{\delta}{\delta J'} \overleftarrow{K} i\Delta^{+} \overrightarrow{K} \frac{\delta}{\delta J''}\right] \mathcal{S}_0\{J'\} \bar{\mathcal{S}}_0\{J''\} \,\mathrm{d}x \Big|_{\substack{J'=\widehat{J} \\ J''=0}}.$$

Wie sich durch Reihenentwicklung nach Potenzen von J zeigen läßt, folgt aus dieser Gleichung, daß die ähnliche Gleichung, in der beiderseits der Faktor $\exp[-J\Delta_F J/2]$ weggelassen ist, ebenfalls gilt. Damit erhalten wir schließlich unter Benutzung der (A.10) entsprechenden Beziehung für $\mathcal{S}_0\{J\}$ als in φ-Funktionen geschriebene Zusammenfassung des Systems A:

$$(A.11)\quad \mathcal{S}_0\{J\} = 1 - \iint J(x) \Delta^{(-)}(x-x') \boldsymbol{K}_{x'} \frac{\delta}{\delta J'(x')} \mathcal{S}_0\{J'\} \,\mathrm{d}x \,\mathrm{d}x' \Big|_{J'(x')=\widehat{J}(x', x)} -$$

$$- \int J(x) \frac{\delta}{\delta J''(x)} \exp\left[J' \Delta^{(+)} \boldsymbol{K} \frac{\delta}{\delta J''}\right] \exp\left[\frac{\delta}{\delta J'} \overleftarrow{K} i\Delta^{(+)} \overrightarrow{K} \frac{\delta}{\delta J''}\right] \mathcal{S}_0\{J'\} \bar{\mathcal{S}}_0\{J''\} \,\mathrm{d}x \Big|_{\substack{J'=\widehat{J} \\ J''=0}}.$$

RIASSUNTO (*)

Si propone una nuova formulazione delle teorie di campo quantizzate. Partendo da alcuni requisiti generali, si deriva una serie di equazioni che determinano gli elementi di matrice degli operatori di campo e la matrice S. Queste equazioni non contengono costanti di rinormalizzazione ma solo masse sperimentali e parametri d'accoppiamento. Il principale vantaggio rispetto alla formulazione convenzionale è, pertanto, l'eliminazione di tutti i termini divergenti dalle equazioni fondamentali. Sono così eliminati i problemi di rinormalizzazione. La formulazione è, in questo lavoro, ristretta alle teorie che non considerano stati legati stabili. Per semplicità deriviamo le equazioni per le particelle con spin nullo; tuttavia, l'estensione ad altri casi (ad esempio, all'elettrodinamica quantistica) è immediata. Si discutono le soluzioni delle equazioni mediante uno sviluppo in serie di potenze. Esse sono allora identiche alle espressioni rinormalizzate della formulazione convenzionale. Tuttavia, l'applicazione delle equazioni date nel presente lavoro non è limitata alla teoria delle perturbazioni.

(*) *Traduzione a cura della Redazione.*

Mass Reversal and the Universal Interaction (*).

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Summary. — It is assumed that the results of relativistic quantum theories should be invariant under the change in sign of the mass term in the Dirac Equation. The consequences of this principle for the Universal Fermi Interaction are that only mixtures of scalar, pseudoscalar and tensor (S, P, T) interactions or mixtures of vector and axial vector interactions (V, A) are possible. Thus no Fierz interferences are allowed. In order to obtain further restrictions on the Fermi interaction, symmetry principles of the kind proposed by PURSEY are considered. It is found that the only possible interactions are the following mutually excluding ones: $(S+P\pm T)$, $(A\pm V)$, $(S+P\pm T/3)$, (S, P) A, V and T.

In this paper a new invariance principle is introduced for the relativistic quantum theory of fields. It is assumed that the results of such theories should be invariant under the change in sign of the mass term in the Dirac equation. This invariance holds indeed both for Quantum Electrodynamics and Meson Theory. If several Dirac particles interact with several Boson fields this principle introduces selection rules against certain interactions. The consequences for the Universal Fermi Interaction are that only such mixtures as scalar, pseudo-scalar and tensor (S, P, T) interaction *or* mixtures of vector and axial vector interactions (V, A) are possible. If also the interactions of π-mesons with nucleons and with μ-mesons are known we may be led to the exclusion of one of these possibilities.

Finally, the consequence of further imposing a symmetry principle of the kind proposed by PURSEY is examined. It is found that even considering

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other possible symmetry principles the only possible Fermi interactions are the following mutually excluding ones: $(S+P-T)$, $(A-V)$, $(S+P+T/3)$; $(S+P+T)$, $(A+V)$, $(S+P-T/3)$; (S, P), A, V and T.

1. – Invariance of Quantum Electrodynamics Under Mass Reversal.

It is known that all physical predictions in quantum electrodynamics are independent of the sign of the mass m in the Dirac equation:

$$(1) \qquad i\hbar \frac{\partial \psi}{\partial t} = V\psi + c\boldsymbol{\alpha}\cdot(\boldsymbol{p} - e\boldsymbol{A})\,\psi \pm mc^2\beta\psi .$$

Most authors use the positive sign and then the two upper components of ψ (in the usual representation of the Dirac matrices) are the « large » ones. Others use the negative sign and the « large » components are then the lower ones.

This is the same as to say that Quantum Electrodynamics is invariant under the transformation

$$(2) \qquad \begin{cases} m \to -m \\ \psi \to \gamma_5\psi \\ A^\mu \to A^\mu . \end{cases}$$

We shall call this transformation « Mass Reversal », after PEASLEE [1] who used it in a somewhat different manner than in the present paper.

The above result can be stated in a more general way as follows: If we have several spinor fields ψ_r (in what follows ψ_r may be considered as one particle wave functions or as field operators), with masses m_r, in interaction only with the electromagnetic potentials, the equations are invariant under « mass reversal » for *any* of the particles (r_0), say, under the transformation:

$$(3) \qquad \begin{cases} m_{r_0} \to -m_{r_0} ; & m_r \to m_r , & r \neq r_0 , \\ \psi_{r_0} \to \eta_{r_0}\gamma_5\psi_{r_0} ; & \psi_r \to \eta_r\psi_r , & r \neq r_0 ; \end{cases}$$

η_r in these transformations are arbitrary phase factors:

$$|\eta_r|^2 = 1 .$$

In general the phases η_r $(r \neq r_0)$ are not essential.

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(1) D. C. PEASLEE: *Phys. Rev.*, **91**, 1447 (1953).

In what follows ψ_r will refer to the positive particles; the corresponding antiparticle fields

$$\psi_r' = C\bar{\psi}_r ,$$

(say the negative particles), will transform then as

$$\psi_r' \to \eta^* \psi_r' ; \qquad \psi_{r_0}' \to -\eta_{r_0}^* \gamma_5 \psi_{r_0}' ; \qquad r \neq r_0 .$$

2. – Invariance of the Meson Theory.

The Dirac equation for nucleons in interaction with the $\pi^{\pm}$, π_0 meson field φ_i are assumed to be (pseudoscalar theory, pseudoscalar interaction):

$$(4) \qquad i\hbar \frac{\partial \psi}{\partial t} = V\psi + c\boldsymbol{\alpha} \cdot (\boldsymbol{p} - c\boldsymbol{A})\psi + Mc^2 \beta \psi + ig\beta\gamma_5 \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \psi ,$$

where $\boldsymbol{\tau}$ is the isotopic spin operator for the nucleons whose wave operators are

$$\psi = \begin{pmatrix} \psi_P \\ \psi_N \end{pmatrix} .$$

This equation is invariant under both the « mass reversal » transformations:

$$(5) \qquad M \to -M ; \qquad \psi \to \gamma_5 \psi ; \qquad \varphi_i \to -\varphi_i$$

and

$$(6) \qquad M \to -M ; \qquad \psi \to \gamma_5 \tau_3 \psi ; \qquad \varphi_1 \to \varphi_1 , \qquad \varphi_2 \to \varphi_2 , \qquad \varphi_3 \to -\varphi_3 .$$

Here $\psi \to \gamma_5 \tau_3 \psi$ is the same as: $\psi_P \to \gamma_5 \psi_P$, $\psi_N \to -\gamma_5 \psi_N$.

Then we see that if the π-meson interacts (with pseudoscalar coupling) also with another spinor χ (say the $\mu^{\pm}$, ν fields), then, in opposition to the case of the electromagnetic fields, the equation for

$$\chi = \begin{pmatrix} \psi_\mu \\ \psi_\nu \end{pmatrix} ,$$

cannot be kept in general unchanged just by allowing phase factors changes for ψ_μ and ψ_ν. We can overcome this difficulty by assuming that this inter-

action (which leads to the $\pi \to \mu$ decay) is not an elementary interaction (²) but a secondary one. Another possibility is to assume that only charged π-mesons interact with the χ field (consistently with the experimentally observed cases). In this case the phase factor change for the μ and ν fields should be the same, if we use transformation (6) for the mass reversal of the nucleon field, or they should have opposite sign if we use (5):

$$(5a) \qquad \psi_\mu \to \eta\psi_\mu\,; \qquad \psi_\nu \to -\eta\psi_\nu\,,$$

$$(6a) \qquad \psi_\mu \to \eta\psi_\mu\,; \qquad \psi_\nu \to \eta\psi_\nu\,.$$

Now the referred interaction of (μ, ν) with the $\pi^\pm$ field will not be invariant under mass reversal of the μ field unless a similar transformation is also performed for the ν field; this implies the assumption that μ and ν form *one* field in the sense that (P, N) form the nucleon field. In this paper we shall assume, for the sake of simplicity, that the neutral particle produced in the $\pi \to \mu$ decay is not a neutrino but a particle μ_0 with small mass:

$$\pi \to \mu + \mu_0'\,.$$

So the field $\chi = \begin{pmatrix} \psi_\mu \\ \psi_{\mu_0} \end{pmatrix}$ will transform under mass reversal in a way similar to the field ψ, as given by (5), (6).

The extension of the above results to other forms of interaction is straightforward. In the case of several fermions interacting with several boson fields, restrictions of the type above referred to and selection rules for elementary interactions (similar to the exclusion of interactions of μ and ν with π_0 field) will appear.

3. – Invariance of the Universal Fermi Interaction.

The well known fact that the interactions which lead to β-decay, μ-decay and μ-capture have practically the same strenght leads to the assumption that the Fermi interaction is Universal for particles of spin $\frac{1}{2}$ and thus is symmetrical in all particles. If all particles are considered different the complete

(²) A similar situation occurs for the interaction of nucleons with the electromagnetic field. Here, if we introduce in the Hamiltonian a Pauli term $a\beta F_{\mu\nu}\gamma^{\mu\nu}$ together with the usual term i.e. $\beta\gamma^\mu A_\mu$ (for protons) we find that the theory cannot be made invariant under mass reversal. If we start, however, from a theory without the Pauli term but involving a meson field (in a way invariant under mass reversal) it is known that the Pauli term may result as a secondary effect.

symmetry ([3]) leads to the Wigner-Critchfield interaction ($S - A - P$) which is unsatisfactory in many respects. In particular this interaction does not obey the mass reversibility principle. If we assume that all particles appear in pairs, such as (P, N), (μ, μ_0), (e, ν) then the symmetry between these (pair) fields does not introduce any restriction. We shall adopt this point of view. So we assume the interaction schemes:

(7*a*) $\quad P \to N + e^+ + \nu' \qquad (\beta \text{ decay}),$

(7*b*) $\quad \mu^+ \to \mu_0 + e^+ + \nu' \qquad (\mu \text{ decay}),$

(7*c*) $\quad P + \mu^- \to N + \mu_0' \qquad (\mu\text{-capture}),$

(7*d*) $\quad \pi^+ \to \mu^+ + \mu_0' \qquad (\pi\text{-decay}).$

The mass reversal transformation should be made simultaneously for both particles in a pair, as in (5), (6).

If, now, we assume that the theory should be invariant under mass reversal for each particle (pair) we are led to the following result:

The only possible forms of Fermi interaction which are invariant under mass reversal are *either* a linear combination of *vector and pseudovector* interactions, or a linear combination of *Scalar, Tensor and Pseudoscalar* interactions, in the usual ordering of β-decay interaction $\bar{\psi}_P \psi_N \bar{\psi}_e \psi_\nu$.

Both cases are consistent with transformations of the types:

a) $\qquad M_P \to - M_P ; \qquad M_N \to - M_N ; \qquad \underset{N}{\psi_P} \to \gamma_5 \underset{N}{\psi_P},$

the other pairs of particles which interact with (PN) transforming with *equal* phases for (V, A) interactions, and with phases of *opposite* signs for (S, P, T) interactions (such as in (5*a*), (6*a*)).

b) $\qquad \underset{N}{M_P} \to - \underset{N}{M_P}, \qquad \underset{N}{\psi_P} \to \pm \gamma_5 \underset{N}{\psi_P},$

the other pairs of particles transforming with equal phases for (S, P, T) and with opposite phases for (V, A) interactions.

The extension to the case of non-Universal interaction is immediate. In this case we find that *all* Fermi interactions which correspond to the processes (7*a*, *b*, *c*) should be of the same type (either (V, A) or (S, P, T)) the only arbitrariness being in the coefficients of the linear combinations.

We see that the consequence of the invariance under mass reversal is to exclude interactions which lead to Fierz interferences ([4]), which appear when

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([3]) C. N. YANG and J. TIOMNO: *Phys. Rev.*, **79**, 495 (1950).

([4]) M. FIERZ: *Zeits. Phys.*, **104**, 553 (1947).

interactions of the group (A, V) are mixed with those of (S, P, T). This could be seen from the beginning by observing that these interference terms, say in β-decay, are linear in the mass of the electron; so they change sign under mass reversal for the (e, ν) field.

Now if we consider simultaneously the π-nucleon interaction and those which lead to processes (7) we may be led to a choice between (S, P, T) and (V, A) theories. Indeed, if we assume pseudoscalar coupling of π-mesons with both (P, N) and (μ, μ_0) then the transformations used for (P, N) imply the relations between the phases of (μ, μ_0) given by (5*a*) and (6*a*) which are correct, for the invariance of the Fermi interaction, only for the (S, P, T) theory. So a definite knowledge of the π-meson interactions may lead to choice in favour of (S, P, T) or (A, V) Fermi interaction.

4. – Unique Determination of the Universal Interaction.

Now that we are left with only (S, P, T) and (A, V) mixtures, we may attempt to add a further principle to obtain the coefficients of the linear combination. A natural assumption is to impose symmetry among the several particles. The complete symmetry which led to the Wigner-Critchfield theory is known to be unsatisfactory. Less strong symmetry properties have already been proposed by several authors. Thus PURSEY [5] has proposed to assume symmetry between the anticommuting operators for the neutral particles (and then also for the charged ones). So he is led (in the usual ordering of β-decay interaction $\bar{\psi}_P \psi_N \bar{\psi}_e \psi_\nu$), to a linear combination of $(S + P - T)$, $(A - V)$ and $(S - A - P)$. Now, if we impose invariance under mass reversal we are restricted to *either* $(S + P - T)$ or $(A - V)$. A final selection between them could result as referred in sec. 3.

This is, however, correct only if we assume *anticommutation* not only between the wave functions for two particles of the same pair (P, N), (μ, μ_0), (e, ν) as it is necessary for the Hamiltonian formalism but also between *particles in different pairs*. If we suppress the last restriction, say, if we assume that the wave operators for two particles in different pairs *commute* (say N and ν) then the interaction which is symmetrical between neutral particles is now a linear combination of [1] $(S + P + T/3)$ and $(2S - 2P + A + V)$. Thus the principle of mass reversibility leads us to the unique possibility, $(S + P + T/3)$.

Pursey's symmetry principle can be expressed in a formal way by stating that the Universal Fermi interaction Hamiltonian is given by

$$H = \sum_{r,s} \sum_{K} G^K O^K_{\alpha\beta} O^K_{\gamma\delta} (\overline{\psi^r_\alpha}\, \overline{\psi'^s_\gamma} + \overline{\psi'^s_\alpha}\, \overline{\psi^r_\gamma})(\varphi^r_\beta \varphi'^s_\delta + \varphi'^s_\beta \varphi^r_\delta) + \text{c. c.} \tag{8}$$

[5] D. PURSEY: *Physica*, **18**, 1017 (1952).

where ψ_r, φ_r are annihilation operators for the positive and neutral particles of the r-th kind. ψ' and φ' are the charge conjugate operators. O_K represent the 5 kinds of covariant operators. According to whether ψ_r and ψ'_s anticommute or commute, we obtain Pursey's result or the other one referred to above.

Another possible symmetry principle is that of « symmetry between particles and between antiparticles » (or between emitted particles and between absorbed particles). In this case we write:

$$H = \sum_{r,s} \sum_{K} G_K O^K_{\alpha\beta} O^K_{\gamma\delta} (\overline{\psi^r_\alpha}\, \overline{\varphi^s_\gamma} + \overline{\varphi^s_\alpha}\, \overline{\psi^r_\gamma})(\varphi^r_\beta \psi^s_\delta + \psi^s_\beta \varphi^r_\delta) + \text{c. c.}\,. \tag{9}$$

Here we find that if ψ_r anticommutes with φ_s then H is a linear combination of $(S+P+T)$, $(A+V)$ and $(S-P-A)$ when expressed in the usual β-decay order. If ψ_r and φ_s commute then H is a linear combination of $(S+P-T/3)$ and $(2S-2P+A-V)$.

Thus, if we impose again the mass reversal invariance we are restricted to the 3 unique possibilities: $(S+P+T)$, $(A+T)$, in the anticommuting case, and $(S+P-T/3)$ in the commuting.

Finally, we could impose « symmetry between particles in the same pair »:

$$H = \sum_{r,s} \sum_{K} G^K O^K_{\alpha\beta} O^K_{\gamma\delta} (\overline{\psi^r_\alpha}\, \overline{\varphi'^r_\beta} \pm \varphi'^r_\alpha\, \overline{\psi^r_\beta})(\varphi^s_\gamma \psi'^s_\delta \pm \psi'^s_\gamma \varphi^s_\delta) + \text{c. c.}\,. \tag{10}$$

Here the operators O^K are the same as in (8), (9) but multiplied by the matrix C.

In this case, as φ_r and φ'_r *anticommute* necessarily, we find that H is of the type (S, P, A) if the plus sign is taken in (10), and (V, T) if we use the minus sign [6]. So the mass reversibility principle leads in this case to the mutually excluding possibilities: (S, P), A, V and T. In this case β^+ and β^- emission would be completely symmetrical [6] except for the Coulomb distortion. These interactions are, however, known to be unsatisfactory for β-decay.

[6] S. R. de Groot and H. A. Tolhoek: *Physica*, **16**, 456 (1950).

RIASSUNTO (*)

Si fa l'ipotesi che i risultati delle teorie quantiche relativistiche siano invarianti rispetto al cambiamento di segno del termine di massa nell'equazione di Dirac. In conseguenza di questo principio, solo miscele di interazioni fra scalari, pseudoscalari e tensori (S, P, T) o miscele di interazioni fra vettori e vettori assiali (V, A) sono possibili nell'interazione universale di Fermi. Pertanto non sono permesse interferenze di Fierz. Per ottenere ulteriori restrizioni all'interazione di Fermi, si considerano principi di simmetria del genere proposto da Pursey. Si trova che le sole interazioni possibili sono le seguenti che si escludono reciprocamente: $(S+P\pm T)$, $(A\pm V)$, $(S+P\pm T/3)$, (S, P), A, V e T.

(*) *Traduzione a cura della Redazione.*

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Interference Effects in the Total Neutron Scattering Cross-Section of Crystals.

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Summary. — A general discussion is given for the interference effects in the total neutron scattering cross-section of crystals. The limiting cases of short and long wavelengths are examined in detail. The inadequacy of the Debye approximation for quantitative calculation of the long wavelength interference term is emphasized.

1. – Introduction.

The total scattering cross-section σ per nucleus of a crystal consisting of a single type of atom is usually written as a sum of two parts

$$\sigma = 4\pi\langle a\rangle^2 S_c + 4\pi\left\{\langle a^2\rangle - \langle a\rangle^2\right\} S_i \,. \tag{1.1}$$

Here a is the nuclear scattering length and $\langle\ \rangle$ denotes averaging over spin and isotopic composition, the mass difference between the isotopes being neglected. The quantities S_c and S_i depend on the wavelength of the incident neutron, the crystal temperature and, in general, on the orientation of the crystal (1).

The first term in (1.1) is called the coherent and the second term the incoherent cross-section of the crystal. The quantities $4\pi\langle a^2\rangle$ and $4\pi\left\{\langle a^2\rangle - \langle a\rangle^2\right\}$

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(1) For literature see G. Placzek and L. Van Hove: *Phys. Rev.*, **93**, 1207 (1954).

are, respectively, the total bound coherent and incoherent cross-sections of the nucleus.

Introducing

$$\delta S = S_c - S_i \,, \tag{1.2}$$

we may rewrite (1.1)

$$\sigma = 4\pi\langle a^2\rangle S_i + 4\pi\langle a\rangle^2 \delta S \,. \tag{1.3}$$

The first term in (1.3) represents the scattering from a single nucleus in the crystal and the second term the interference effect.

S_i may be evaluated rapidly by expressing the neutron wavelength in terms of the neutron energy and thereupon expanding S_i in powers of the neutron mass ([2]). This expansion is usually more practical than the expression of S_i as a sum over cross-sections S_{il} for the production or destruction of l phonons (phonon expansion), which often converges very poorly, particularly for high temperatures or short wavelengths. The former expansion, however, does not in general apply to δS.

For single crystals, δS is a highly singular function of the wave vector $\boldsymbol{k}_0$ of the incident neutron. The main singularities occur, both in the elastic and inelastic parts of δS, at the values of $\boldsymbol{k}_0$ satisfying the Bragg condition

$$\tau^2 + 2\boldsymbol{k}_0 \cdot \boldsymbol{\tau} = 0 \,, \tag{1.4}$$

where $\boldsymbol{\tau}$ is a lattice vector of the reciprocal lattice (*). For polycrystals the order of these singularities is greatly reduced; here δS remains finite throughout and has discontinuities at $k_0 = \tau/2$, which are approached with a logarithmically infinite derivative. The size of these discontinuities decreases as $\exp[-ck_0^2]$; they are thus of little importance for large k_0.

The contributions to δS and S_i of the individual processes characterized by the production or destruction of l phonons with wave vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_l$, accompanied by a change in the wave vector of the neutron from the initial value $\boldsymbol{k}_0$ to a final value $\boldsymbol{k}$, differ by the factor

$$\sum_{\boldsymbol{R}}{}' \exp i\{\boldsymbol{\varkappa} + \sum_{i=1}^{l} \varepsilon_i \boldsymbol{q}_i\}\boldsymbol{R} \,,$$

due to the interference between the waves scattered from different nuclei.

([2]) G. PLACZEK: *Phys. Rev.*, **93**, 897 (1954).

(*) We define the reciprocal lattice as the set of vectors $\boldsymbol{\tau}$, whose inner products with the lattice vectors $\boldsymbol{R}$ of the crystal lattice are integer multiples of 2π.

The sum extends over the lattice vectors $\boldsymbol{R}$ with omission of $R = 0$. $\varepsilon_i = \pm 1$ where the upper (lower) sign holds for destruction (production) of the phonon i, and

$$\boldsymbol{\varkappa} = \boldsymbol{k}_0 - \boldsymbol{k} \,. \tag{1.5}$$

By transformation into Fourier space this factor may be expressed by

$$\sum_{\boldsymbol{R}}{}' \exp i\Big\{\boldsymbol{\varkappa} + \sum_{i}^{l} \varepsilon_i \boldsymbol{q}_i\Big\}\boldsymbol{R} = -1 + \frac{(2\pi)^3}{v_0} \sum_{\boldsymbol{\tau}} \delta\Big\{\boldsymbol{\varkappa} + \Big\{\sum_{i=1}^{l} \varepsilon_i \boldsymbol{q}_i\Big\} + \boldsymbol{\tau}\Big\}, \tag{1.6}$$

where v_0 is the volume per atom (*). Because of the factor (1.6) the phonon expansion of δS will converge far more rapidly than the phonon expansion of S_i. While similar statements may be found in the literature ([3,4]), they have not as yet been substantiated by general estimates. For short wavelength such general estimates can be readily obtained from the evaluation of δS carried out in a previous paper ([5]). In the following section we discuss the case of short wavelength and show that for a polycrystal δS is here already represented by its elastic part δS_0 with a relative error of order $(u_0/d)^2$, (where u_0 is the root mean square displacement of a nucleus from its equilibrium position and d the lattice constant) and by the sum $\delta S_0 + \delta S_1$ of its elastic and one phonon parts with a relative error of order $(u_0/d)^4$. After some remarks on the case of general wavelength in section **3**, we examine, in sections **4** and **5**, the opposite limiting case of long wavelength. This case turns out to be more complicated and it has not been possible to reach conclusions as definite as in the short wavelength region. The order of magnitude of the ratio $\delta S/S_i$ has been determined by a half-numerical method based on the Debye approximation. It establishes at the same time that strong fluctuations of this ratio should be expected from substance to substance. It is also made clear that when quantitative calculations are concerned, the Debye approximation is much less adequate for coherent effects such as δS than for the calculation of the incoherent cross-section.

The present paper makes rather extensive use of the formal theory of scattering by crystals. A less formal account of the present state of the problem on the basis of a review of experimental results will be found in a forthcoming paper by SQUIRES.

(*) In order to simplify notation we consider here and in the following only Bravais lattices, with one particle per cell.

([3]) A. AKHIEZER and L. POMERANCHUK: *Journ. Phys. USSR*, **11**, 167 (1947).

([4]) G. L. SQUIRES: *Proc. Roy. Soc.* (*London*), A **212**, 192 (1952).

([5]) G. PLACZEK, B. R. A. NIJBOER and L. VAN HOVE: *Phys. Rev.*, **82**, 392 (1951). In the following, this paper will be referred to as A.

2. – Short Wavelenghts.

For short neutron wavelengths, the incoherent cross-section of a polycrystal (*) is given by the asymptotic expression ([6,7])

$$\langle S_i \rangle = \left(1 + \frac{1}{\mu}\right)^2 \left(1 + \frac{K_{\mathrm{Av}}}{3\mu E_0}\right) - \frac{1}{32}\,\frac{C_{\mathrm{Av}}}{\mu E_0^3}\,; \tag{2.1}$$

μ is the ratio of nuclear to neutron mass, E_0 the neutron energy, K_{Av} the average kinetic energy of the nucleus and C_{Av} is related to the average square of the binding force acting on the nucleus. Explicit expressions for the coefficients K_{Av} and C_{Av} as functions of the temperature have been given in ref. ([6]) in the Debye approximation.

For heavy nuclei (2.1) begins to hold at a neutron energy slightly larger than the Debye temperature and for lighter nuclei at somewhat higher neutron energies. An asymptotic expression for the interference term $\langle \delta S \rangle$, valid for neutron wavelengths satisfying the condition $k_0 u_0 > 1$ (u_0 root mean square nuclear displacement in any direction) has been derived in A on the basis of the static approximation which assumes zero neutron mass, infinite neutron energy and finite neutron wavelength (**). It is given by

$$\langle \delta S \rangle = \frac{\lambda^2}{8\pi v_0^{\frac{2}{3}}}\, I\,, \tag{2.2}$$

where λ is the neutron wavelength and I a numerical coefficient, given by

$$I = I^{(0)} - I^{(1)}\,\frac{u_0^2}{v_0^{\frac{2}{3}}} - I^{(2)}\,\frac{u_0^4}{v_0^{\frac{4}{3}}} - \ldots\,. \tag{2.3}$$

$I^{(0)}$ is the limiting value of I for $u_0 \to 0$, shown in A to be represented by ([+]):

$$I^{(0)} = 3 - v_0^{\frac{2}{3}} \sum_R{}' \frac{\exp\left[-\pi v_0^{-\frac{2}{3}} R^2\right]}{\pi R^2} - \frac{2\pi}{v_0^{\frac{1}{3}}} \sum_\tau{}' \frac{\varphi(\tau v_0^{\frac{1}{3}}/2\pi^{\frac{1}{2}})}{\tau}\,, \tag{2.3a}$$

(*) In the following the quantities S_i and δS refer to a single crystal and $\langle S_i \rangle$ and $\langle \delta S \rangle$ to a polycrystal.

([6]) G. PLACZEK: *Phys. Rev.*, **86**, 377 (1952).

([7]) G. C. WICK: *Phys. Rev.*, **94**, 5, 1228-1242 (1954).

(**) For a more detailed definition of this approximation see ref. ([6]) and for a discussion of the validity of its use for the calculation of interference terms at short neutron wavelength, refs. ([6]) and ([7]) and section 3.

([+]) In A this quantity was denoted by I_0. In the present paper we reserve lower indices for the phonon contributions, cf. (2.15).

where

$$\varphi(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp[-y^2]\,\mathrm{d}y.$$

The remaining terms in (2.3) represent the effect of thermal and zero point motion and amount, in general, to very small corrections only. They are hardly of practical importance and we will have to consider them in the following merely in order to establish the degree of convergence of the phonon expansion of $\langle \delta S \rangle$.

At neutron wavelengths short enough for $\langle \delta S \rangle$ to be represented by Eq. (2.2), $(k_0 u_0 > 1)$, the decomposition of Eq. (2.1) for $\langle S_i \rangle$ into l-phonon cross-sections is rather complicated. If $4k_0^2 u_0^2 \gg 1$, then the number of strongly contributing terms in the phonon expansion of $\langle S_i \rangle$ is of the order $4k_0^2 u_0^2$ and the elastic and low phonon cross-sections are each of the order $1/4k_0^2 u_0^2$. On the other hand it will now be shown how much more rapidly the phonon expansion of $\langle \delta S \rangle$ converges. For this purpose we have to review briefly the derivation of (2.2) and (2.3).

If the differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ is expressed by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \langle a \rangle^2 s_c + \left\{ \langle a^2 \rangle - \langle a \rangle^2 \right\} s_i = \langle a^2 \rangle s_i + \langle a \rangle^2 \delta s\,, \qquad \delta s = s_c - s_i\,, \tag{2.4}$$

then

$$\delta S = \frac{1}{4\pi} \int \delta s\, \mathrm{d}\Omega\,,$$

and, in static approximation, as shown in A,

$$\langle \delta S \rangle = \frac{1}{8\pi k_0^2} \int_{2k_0} \delta s\, \frac{\mathrm{d}\varkappa}{\varkappa}\,. \tag{2.5}$$

The integration variable $\boldsymbol{\varkappa}$ is defined by (1.5) and the suffix $2k_0$ indicates that the integration extends over a sphere of radius $2k_0$.

In static approximation, δs is given by (*)

$$\delta s = \sum{}' \exp[i\boldsymbol{\varkappa} \cdot \boldsymbol{R} - \varkappa^2 \gamma_\varkappa(\boldsymbol{R})] - \frac{(2\pi)^3}{v_0}\, \delta(\boldsymbol{\varkappa})\,. \tag{2.6}$$

The summation extends over all lattice vectors $\boldsymbol{R}$ with the exception of

(*) Apart from a trivial generalization, Eq. (2.6) is identical with the expression derived in A.

the origin and $\gamma_\varkappa$ is defined by

$$\gamma_\varkappa(\boldsymbol{R}) = \alpha_\varkappa(\boldsymbol{R}) - \beta_\varkappa(\boldsymbol{R}) , \tag{2.7}$$

where $\alpha_\varkappa$ is the mean square displacement of a nucleus in the direction of $\boldsymbol{\varkappa}$ and $\beta_\varkappa$ the average of the product of the components $u_\varkappa$, in the direction of $\boldsymbol{\varkappa}$, of the displacement vectors of two different nuclei. Explicitly:

$$\beta_\varkappa(\boldsymbol{R}) = \langle u_\varkappa(0) u_\varkappa(\boldsymbol{R}) \rangle = \tag{2.8}$$

$$= \frac{v_0}{(2\pi)^3} \sum_j \int e_\varkappa^2 \frac{\hbar}{2M\omega} \left\{ \frac{\exp[i\boldsymbol{q}\cdot\boldsymbol{R}]}{\exp[\hbar\omega/T] - 1} + \frac{\exp[-i\boldsymbol{q}\cdot\boldsymbol{R}]}{1 - \exp[-\hbar\omega/T]} \right\} \mathrm{d}\boldsymbol{q} ,$$

$$\alpha_\varkappa = \beta_\varkappa(0) . \tag{2.9}$$

Here $\boldsymbol{q}$ is the wave vector of a sound wave, M the nuclear mass; the frequencies ω and the polarization vectors $\boldsymbol{e}$ are functions of the polarization index j and wave vector $\boldsymbol{q}$. The integration extends over one cell in $\boldsymbol{q}$-space.

Eq. (2.8) may also be written

$$\beta_\varkappa(\boldsymbol{R}) = \frac{v_0}{(2\pi)^3} \sum_j \int e_\varkappa^2 \frac{E}{M\omega^2} \cos \boldsymbol{q}\cdot\boldsymbol{R} \, \mathrm{d}\boldsymbol{q} , \tag{2.10}$$

where

$$E = \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2T} ,$$

is the energy of a lattice vibration of frequency $\omega(\boldsymbol{q}, j)$ at temperature T. From the definition of $\beta_\varkappa$ as correlation coefficient it follows that

$$-\alpha_\varkappa \leqslant \beta_\varkappa(\boldsymbol{R}) \leqslant \alpha_\varkappa ,$$

which is also seen from (2.10) and (2.9). For large R, $\beta_\varkappa(\boldsymbol{R})$ tends to zero as $1/R$ for all finite temperatures and as $1/R^2$ for $T = 0$ ([8,5]).

The contributions δs_l of the individual l-phonon processes are obtained by expanding δs in powers of β. With (2.7), Eq. (2.6) may be written

$$\delta s = \exp[-\varkappa^2\alpha_\varkappa] \left\{ \sum_{\boldsymbol{R}}{}' \exp[i\boldsymbol{\varkappa}\cdot\boldsymbol{R} + \varkappa^2\beta_\varkappa(\boldsymbol{R})] - \frac{(2\pi)^3}{v_0} \delta(\boldsymbol{\varkappa}) \right\} . \tag{2.6a}$$

([8]) R. E. PEIERLS: *Ann. Inst. H. Poincaré*, **5**, 177 (1935).

For the elastic contribution we have thus

$$\delta s_0 = \exp[-\varkappa^2\alpha_\varkappa]\left\{\sum_{\boldsymbol{R}}{}'\exp[i\boldsymbol{\varkappa}\boldsymbol{R}] - \frac{(2\pi)^3}{v_0}\,\delta(\boldsymbol{\varkappa})\right\} = \tag{2.11}$$

$$= \exp[-\varkappa^2\alpha_\varkappa]\left\{-1 + \frac{(2\pi)^3}{v_0}\sum_{\boldsymbol{\tau}}{}'\delta(\boldsymbol{\varkappa}+\boldsymbol{\tau})\right\}.$$

The contribution of processes in which l phonons participate ($l > 0$) is given by

$$\delta s_l = \frac{\boldsymbol{\varkappa}^{2l}\exp[-\varkappa^2\alpha_\varkappa]}{l!}\sum_{\boldsymbol{R}}{}'\,\beta_\varkappa^l(\boldsymbol{R})\exp[i\boldsymbol{\varkappa}\cdot\boldsymbol{R}]. \tag{2.12}$$

For our purposes it will not be necessary to discuss the representation of this sum in τ-space.

From the discussion of the behaviour of δs for large $\varkappa$ given in A it results that for $k_0 u_0$ larger than about 1, the integral in (2.5) may be extended over all space. $\langle \delta S\rangle$ is then given by (2.2), with I defined by

$$I = \frac{v_0^{\frac{2}{3}}}{4\pi^2}\int \delta s\,\frac{\mathrm{d}\boldsymbol{\varkappa}}{\varkappa}\,. \tag{2.13}$$

Introduction of (2.6) into (2.13) yielded, under neglection of the dependence of $\gamma_\varkappa$ on the direction of $\boldsymbol{\varkappa}$

$$I = I_0 - \frac{2v_0^{\frac{2}{3}}}{\pi}\left\{\sum_{\boldsymbol{R}}{}'\frac{\gamma(\boldsymbol{R})}{R^4} + 6\sum_{\boldsymbol{R}}{}'\frac{\gamma^2(\boldsymbol{R})}{R^6}\right\}. \tag{2.14}$$

This defines the coefficients I_1 and I_2 in (2.3). The results of the evaluation of the first sum in (2.14) for high temperatures in the Debye approximation are also given in A.

It can further be shown that the expression in the curved brackets represents the first two terms of a semiconvergent expansion, given by

$$\tfrac{1}{2}\sum_{m=1}^{m_{\max}}\frac{(2m)!}{m!}\sum_{\boldsymbol{R}}{}'\frac{\gamma^m(R)}{R^{2m+2}}\,.$$

In order to obtain the contribution of the individual phonon cross-sections to I, one could insert (2.11) and (2.12) into (2.13). The result is obtained far more rapidly, however, by simply expanding (2.13) in powers of β. For the elastic contribution I_0 one has thus

$$I_0 = I^{(0)} - \frac{2}{\pi}\,\frac{\alpha}{v_0^{\frac{2}{3}}}\left\{\sum_{\varrho}{}'\,\varrho^{-4} + 6\frac{\alpha}{v_0^{\frac{2}{3}}}\sum_{\varrho}{}'\,\varrho^{-6} + \ldots\right\}, \tag{2.15}$$

where

$$\rho = \frac{R}{v_0^{\frac{1}{3}}}.$$

The one phonon contribution

$$I_1 = \frac{2}{\pi} v_0^{\frac{8}{3}} \left\{ \sum_{\boldsymbol{R}}{}' \frac{\beta(\boldsymbol{R})}{R^4} - 2\alpha \sum_{\boldsymbol{R}}{}' \frac{\beta(\boldsymbol{R})}{R^6} + \ldots \right\} \tag{2.16}$$

and

$$I_2 = -\frac{12}{\pi} v_0^{\frac{8}{3}} \left\{ \sum_{\boldsymbol{R}}{}' \frac{\beta^2(\boldsymbol{R})}{R^6} + \ldots \right\}. \tag{2.17}$$

Eq. (2.15) shows that the elastic contribution I_0 differs from both I and $I^{(0)}$ only by terms of order $(u_0/d)^2$. Thereby it is shown that the interference term $\langle \delta S \rangle$ is almost entirely due to the elastic cross-section. From (2.16) and (2.17) it is seen that I_1 is of order $(u_0/d)^2$ and I_2 of order $(u_0/d)^4$. While the general estimate $I_l = 0((u_0/d)^{2l})$ will not hold for high l because of the asymptotic nature of the expansion (2.14), it is nevertheless true that for low l_0

$$\sum_{l_0}^{\infty} I_l = 0((u_0/d)^{2l_0}) .$$

In the wavelength region where $\langle \delta S \rangle$ and $\langle S_i \rangle$ are represented by (2.2) and (2.3), both $\langle \delta S \rangle$ and $\langle \delta S_i \rangle / \langle S_i \rangle$ are of order $(\lambda/d)^2$. Denoting the l-phonon contributions to S_i and δS by S_{il} and δS_l respectively, it results from the foregoing discussion that for $(2k_0u_0)^2 \gg 1$ and low l, S_{il} is of order $(2k_0u_0)^{-2}$ and δS_l of order $(\lambda/d)^2(u_0/d)^{2l}$. Accordingly the relative size $\langle \delta S_0 \rangle / \langle S_0 \rangle$ of the interference term in the elastic cross-section is of order $(u_0/d)^2$ and $\langle \delta S_l \rangle / \langle S_{il} \rangle$ for low l of order $(u_0/d)^{2l+2}$.

These results may be interpreted as follows. The interference term δs in the differential cross-section, from which $\langle \delta S \rangle$ results by integration over $d\boldsymbol{\varkappa}$ according to (2.5), is a function of $\boldsymbol{\varkappa}$ which oscillates with the period of the reciprocal lattice. The region in $\varkappa$-space which effectively contributes to the integral is determined by the condition $\varkappa < 2k_0$ and, because of the Debye-Waller factor $\exp[-\varkappa^2\alpha_\varkappa]$ multiplying the right hand side of (2.6a), by $\varkappa \lesssim 2/u_0$. For $k_0 > 1/u_0$ the second condition is the effective one, so that here the accessible volume in $\varkappa$-space is of order u_0^{-3}.

Introducing the ratio F of the accessible volume to the volume of a cell, we see that F is of order $(d/u_0)^3$ and thus $\langle \delta S_0 \rangle / \langle S_{i0} \rangle$ of order $F^{-\frac{2}{3}}$, $\langle \delta S_i \rangle / \langle S_{il} \rangle$ of order $F^{-\frac{1}{3}}$, etc. This rapid decrease with increasing l is due to the decrease of the integrated factor (1.6) with increasing number of integrations over $d\boldsymbol{q}_i$.

We conclude this section by remarking that the dependence of $\gamma_\varkappa$ on the direction of $\boldsymbol{\varkappa}$, which was neglected in the derivation of (2.14) can be readily taken into account, with the result that in (2.14) γ has to be replaced by $\sum_i (4\nu_i^2 - 1)\gamma_i$ and γ^2 by

$$\sum_{ij} \gamma_i\gamma_j\left\{\tfrac{1}{3} - 4\nu_i^2 + 16\nu_i^2\nu_j^2\right\} + \sum_i \gamma_i(\tfrac{2}{3} - 8\nu_i^2)\,, \tag{2.18}$$

where γ_i $(i = 1, 2, 3)$ are the principal values of the symmetric tensor $\boldsymbol{\gamma}(\boldsymbol{R})$ the components γ_{xy} of which are defined by (2.7)-(2.9) with $e_\varkappa^2$ replaced by $e_x e_y$, and ν_i are the components of the unit vector $\nu = \boldsymbol{R}/R$ in the direction of the principal axes of γ.

The corresponding changes in (2.15) and (2.16) and (2.17) are thereby defined. For cubic crystals, in particular, α is a scalar. In this case (2.15) remains unchanged, while in (2.16) and (2.17) β is replaced by $\sum_i (4\nu_i^2 - 1)\beta_i$ and β^2 by (2.18) with β_i and β_j for γ_i and γ_j.

Our general conclusions are in no way affected by the tensor character of γ.

3. – Remarks on the Case of General Wavelength.

We introduce the differential cross-section $\mathrm{d}\sigma/\mathrm{d}\boldsymbol{k}$ per unit volume in the space of the wave vector $\boldsymbol{k}$ of the scattered neutron by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\boldsymbol{k}} = 4\pi\langle a^2\rangle \frac{\mathrm{d}S_i}{\mathrm{d}\boldsymbol{k}} + 4\pi\langle a\rangle^2 \frac{\mathrm{d}\,\delta S}{\mathrm{d}\boldsymbol{k}}\,. \tag{3.1}$$

In static approximation

$$\frac{\mathrm{d}\,\delta S}{\mathrm{d}\boldsymbol{k}} = \frac{1}{2\pi k_0}\,\delta(k^2 - k_0^2)\,\delta s(\boldsymbol{\varkappa}) = \frac{1}{2\pi k_0}\,\delta(\varkappa^2 - 2\boldsymbol{k}_0\cdot\boldsymbol{\varkappa})\,\delta s(\boldsymbol{\varkappa})\,. \tag{3.2}$$

Introducing (2.6) and expressing the δ-function by its Fourier representation

$$\delta(\varkappa^2 - 2\boldsymbol{k}_0\cdot\boldsymbol{\varkappa}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[i(\varkappa^2 - 2\boldsymbol{k}_0\cdot\boldsymbol{\varkappa})t'\right]\mathrm{d}t'\,, \tag{3.3}$$

we obtain

$$\frac{\mathrm{d}\,\delta S}{\mathrm{d}\boldsymbol{k}} = \frac{1}{4\pi^2 k_0}\int_{-\infty}^{\infty}\mathrm{d}t'\left\{\sum_{\boldsymbol{R}}{}' \exp\left[i\boldsymbol{\varkappa}(\boldsymbol{R} - 2\boldsymbol{k}_0 t') - \varkappa^2(\gamma_\varkappa - it')\right] - \frac{(4\pi)^3}{v_0}\,\delta(\boldsymbol{\varkappa})\right\}. \tag{3.4}$$

The advantage of this expression over (2.6) lies in the fact that it lends

itself more readily to generalization. It has been shown ([9-11]), that (3.4) remains valid in the general theory without the restrictions imposed by the static approximation, if γ is defined as a time-dependent quantity, by

$$\gamma_{xy}(\boldsymbol{R}, t) = \alpha_{xy} - \beta_{xy}(\boldsymbol{R}, t) , \tag{3.5}$$

$$t = \frac{2m}{\hbar} t' , \tag{3.6}$$

where $\boldsymbol{\alpha}$ is defined as in section 2, while $\boldsymbol{\beta}(R, t)$ is given by

$$\beta_{xy}(\boldsymbol{R}, t) = \langle u_x(0, 0)\, u_y(\boldsymbol{R}, t)\rangle = \tag{3.7}$$

$$= \frac{v_0}{(2\pi)^3} \sum_j \int e_x \cdot e_y \frac{\hbar}{2M\omega} \left\{ \frac{\exp[i(\boldsymbol{q}\cdot\boldsymbol{R} - \omega t)]}{\exp[\hbar\omega/T] - 1} + \frac{\exp[-i(\boldsymbol{q}\boldsymbol{R} - \omega t)]}{1 - \exp[-\hbar\omega/T]} \right\} \mathrm{d}\boldsymbol{q} ,$$

$$\alpha_{xy} = \langle u_x(0, 0)\, u_y(0, 0)\rangle = \beta_{xy}(0, 0) . \tag{3.8}$$

In order to obtain δS we integrate (3.4) over $\mathrm{d}\boldsymbol{\varkappa}$. This yields

$$\delta S = \int \frac{\mathrm{d}\,\delta S}{\mathrm{d}\boldsymbol{k}} \mathrm{d}\boldsymbol{\varkappa} = \frac{1}{4\sqrt{\pi} k_0} \int_{-\infty}^{\infty} \mathrm{d}t' \left\{ \sum_{\boldsymbol{R}}{}' (\|b\|)^{\frac{3}{2}} \exp\left[-\frac{\boldsymbol{R}'\cdot\boldsymbol{b}\boldsymbol{R}'}{4} \right] - \frac{(4\pi)^{\frac{3}{2}}}{v_0} \right\} . \tag{3.9}$$

Here $\boldsymbol{b}$ is the tensor $[\boldsymbol{\gamma}(\boldsymbol{R}, t) - it]^{-1}$, $\|b\|$ its determinant and

$$\boldsymbol{R}' = \boldsymbol{R} - 2\boldsymbol{k}_0 t' = \boldsymbol{R} - \boldsymbol{V}_0 t , \tag{3.10}$$

where $\boldsymbol{V}_0$ is the velocity of the incident neutron.

The short wavelength results of section 2 are obtained from (3.9) by introducing the variable $x = 2k_0 t'$, averaging over the directions of $\boldsymbol{k}_0$ and thereupon taking the limit $\lim_{k_0 \to \infty} k_0^2 \langle \delta S\rangle$.

For general wavelength direct evaluation of (3.4) is not practical and one has to take recourse to the phonon expansion. Expanding (3.4) in powers of β one has, for cubic symmetry (α scalar)

$$\delta S_0 = \frac{1}{4\sqrt{\pi} k_0} \int_{-\infty}^{\infty} \mathrm{d}t' \left\{ \sum_{\boldsymbol{R}}{}' \frac{\exp[-R'^2/4(\alpha - it')]}{(\alpha - it')^{\frac{3}{2}}} - \frac{(4\pi)^{\frac{3}{2}}}{v_0} \right\} , \tag{3.12}$$

$$\delta S_1 = \sum_{\boldsymbol{R}}{}' \frac{1}{8\sqrt{\pi} k_0} \int_{-\infty}^{\infty} \frac{\exp[-R'^2/4(\alpha - it')]}{(\alpha - it')^{\frac{5}{2}}} \left\{ \mathrm{Tr}(\boldsymbol{\beta}) - \tfrac{1}{2} \frac{\boldsymbol{R}'\cdot\boldsymbol{\beta}\boldsymbol{R}'}{\alpha - it'} \right\} \mathrm{d}t' , \tag{3.13}$$

where $\mathrm{Tr}(\boldsymbol{\beta})$ stands for the trace of β.

([9]) D. A. KLEINMAN: *Thesis Brown University*, 1951.
([10]) R. J. GLAUBER: *Phys. Rev.*, **87**, 189 (1952) and as quoted in ref. ([11]).
([11]) L. VAN HOVE: *Phys. Rev.*, **95**, 249 (1954).

Transformation of (3.12) into τ-space yields immediately

$$(3.14)\quad \delta S_0 = \frac{1}{2\pi k_0}\left\{\frac{(2\pi)^3}{v_0}\sum_{\tau}{}'\exp[-\alpha\tau^2]\,\delta(\tau^2+2\boldsymbol{k}_0\boldsymbol{\tau}) - \right.$$

$$\left. -\int\exp[-\alpha\tau^2]\,\delta(\tau^2+2\boldsymbol{k}_0\cdot\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau}\right\} = \frac{4\pi^2}{k_0 v_0}\sum_{\tau}{}'\exp[-\alpha\tau^2]\,\delta(\tau^2+2\boldsymbol{k}_0\cdot\boldsymbol{\tau}) -$$

$$-\frac{1}{4\alpha k_0^2}\left(1-\exp[-4\alpha k_0^2]\right).$$

This expression for the elastic part of the interference term in a single crystal, showing the δ-singularities for $\boldsymbol{k}_0$ satisfying (1.4), can also be obtained directly from (2.11), since the static approximation is exact for the elastic cross-section.

The representation of δS_1 in τ-space is also relatively simple. In section 4 we shall discuss it, together with (3.13), for the case of long wavelength and mention here only that it shows δS_1 to have logarithmic singularities for $\boldsymbol{k}_0$ satisfying (1.4).

By averaging over the directions of $\boldsymbol{k}_0$, (3.14) goes over into

$$(3.15)\quad \langle\delta S_0\rangle = \frac{1}{k_0^2}\left\{\frac{\pi^2}{v_0}\sum_{\tau}^{0<\tau\leq 2k_0}\frac{\exp[-\alpha\tau^2]}{\tau} - \frac{1}{4\alpha}\left(1-\exp[-4\alpha k_0^2]\right)\right\}.$$

Eq. (3.15) shows the well-known discontinuities of $\langle\delta S_0\rangle$, which occur for $k_0 = \tau/2$ and have the size

$$(3.16)\quad \langle\delta S_0\rangle_+ - \langle\delta S_0\rangle_- = \frac{\pi^2}{v_0}\frac{\exp[-4\alpha k_0^2]}{k_0^3}, \qquad k_0 = \frac{\tau}{2} \neq 0.$$

The logarithmic singularities of δS_1 reduce, by averaging over the directions of $\boldsymbol{k}_0$, to logarithmic singularities in the derivative $\mathrm{d}\langle\delta S_1\rangle/\mathrm{d}k_0$ at $k_0 = \tau/2 \neq 0$.

In section 2 it has been shown that at short wavelength $\langle\delta S\rangle$ is well represented by its elastic part $\langle\delta S_0\rangle$. This will also be the case at somewhat longer wavelengths where the asymptotic expression for $\langle\delta S_0\rangle$ has to be replaced by (3.15). While this approximation neglects finer points such as the large derivative in the immediate neighborhood of $k_0 = \tau/2$, this is not particularly serious for the size of $\langle\delta S\rangle$ as long as $k_0 d \gg 1$. For longer wavelengths. however, the approximation becomes questionable. That it will ultimately have to break down is directly seen from the fact that for k_0 tending to zero, $\langle\delta S_0\rangle$ tends to -1, while the inelastic part of $\langle\delta S\rangle$ increases, for all finite temperatures, as k_0^{-1}. In the following section we shall accordingly examine $\langle\delta S_1\rangle$ in the limiting case $k_0 \to 0$. The significance of this limiting case and the degree of its realizability with experimentally accessible neutron wavelengths have been discussed in a previous paper ([12]).

([12]) G. Placzek and L. Van Hove: *Phys. Rev.*, **93**, 1207 (1954). In the following this paper is referred to as B.

4. – Long Wavelengths.

For $k_0 \to 0$, S_i and δS become independent of the orientation of the crystal. We have, accordingly

$$\langle S_i \rangle = S_i\,, \qquad \langle \delta S \rangle = \delta S\,.$$

We examine the limit $\lim_{k_0 \to 0} (k_0\,\delta S_1)$. This limit is obtained by putting $k_0 = 0$ in the integrand of (3.13), whereby $\boldsymbol{R}'$ goes over into $\boldsymbol{R}$. Hence

$$\lim_{k_0 \to 0} (k_0\,\delta S_1) = \sum_{\boldsymbol{R}}{}' \frac{1}{8\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp\left[-R^2/4(\alpha - it')\right]}{(\alpha - it')^{\frac{5}{2}}} \left\{ \mathrm{Tr}(\boldsymbol{\beta}) - \frac{\frac{1}{2}\boldsymbol{R}\cdot\boldsymbol{\beta}\boldsymbol{R}}{\alpha - it'} \right\} \mathrm{d}t' \,. \tag{4.1}$$

Expressing β by (3.7) and using the relation

$$\int_{-\infty}^{\infty} \frac{\exp\left[-R^2/4(\alpha - it')\right]}{(\alpha - it')^{p}} \exp\left[-ict'\right] \mathrm{d}t' = \tag{4.2}$$

$$= \begin{cases} 0 & \text{for} \quad c \leqslant 0\,, \\ 2\pi \exp\left[-\alpha c\right] \left(\dfrac{2c^{\frac{1}{2}}}{R}\right)^{p-1} J_{p-1}(c^{\frac{1}{2}}R) & \text{for} \quad c \geqslant 0\,, \end{cases}$$

where $p > 1$, $\alpha > 0$, one obtains

$$\lim_{k_0 \to 0} (k_0\,\delta S_1) = \frac{v_0}{(2\pi)^3} \frac{m}{M} \sum_{\boldsymbol{R}}{}' \frac{1}{R} \sum_j \int \frac{\exp\left[-\alpha k^2\right]}{\exp\left[\hbar^2 k^2/2mT\right] - 1} \cdot \tag{4.3}$$

$$\cdot \left\{ \sqrt{\frac{\pi}{2kR}}\, J_{\frac{3}{2}}(kR) - (\boldsymbol{e}\cdot\boldsymbol{\nu})^2 \sqrt{\frac{\pi kR}{2}}\, J_{\frac{5}{2}}(kR) \right\} \exp\left[i\boldsymbol{q}\cdot\boldsymbol{R}\right] \mathrm{d}\boldsymbol{q},$$

where k is a function of $\boldsymbol{q}$ and j given by

$$k^2 = \frac{2m\omega(j\cdot\boldsymbol{q})}{\hbar}\,, \tag{4.3a}$$

and, as in section 2, $\boldsymbol{\nu} = \boldsymbol{R}/R$.

In order to compare δS_1 to S_{i1}, we have to determine $\lim_{k_0 \to 0} k_0 S_{i1}$. This is done by replacing in (4.3) the sum over $\boldsymbol{R}$ by the limit of the summand for $R = 0$ so that

$$\lim_{k_0 \to 0} k_0 S_{i1} = \frac{v_0}{(2\pi)^3} \frac{m}{M} \frac{1}{3} \sum_j \int \frac{\exp\left[-\alpha k^2\right]}{\exp\left[\hbar^2 k^2/2mT\right] - 1} k\, \mathrm{d}\boldsymbol{q}\,. \tag{4.4}$$

International Conference on Elementary Particles and XLI Congress of the Italian Physical Society

PISA 12th-18th JUNE 1955

From June 12th to 18th 1955 an International Conference on Elementary Particles will be held in Pisa, organized by our Society under the auspices and with the support of the International Union of Physics, of the Italian National Research Council, of the Italian Physical Society itself, and of the Town Committee formed expressly to this purpose.

The Conference will be organized in two parallel sections, of which one on *Heavy Mesons and Hyperons* shall have essentially experimental character though including a general report on the theoretical features of the topic, the other, essentially theoretical, will deal with *Quantum Field Theory* and *Pion-Nucleon Interactions*. This last topic will be preceded by a large report on the latest experimental results.

On the last day there will be held a general discussion by the united sections preceded by summarizing reports on the work of both sections.

At the same time the XLI Congress of the Italian Physical Society will take place in Pisa in its customary form.

Secretary General of the whole organization is prof. Marcello Conversi Director of the Physical Institute of the University of Pisa (Piazza Torricelli, 2).

Conferenza internazionale sulle particelle elementari

e

XLI Congresso della Società Italiana di Fisica

PISA 12-18 GIUGNO 1955

Si comunica che nei giorni 12-18 Giugno 1955 si terrà a Pisa una Conferenza internazionale sulle particelle elementari, organizzata dalla nostra Società sotto gli auspici e col contributo dell'Unione Internazionale di Fisica, del Consiglio Nazionale delle Ricerche, della stessa Società Italiana di Fisica e del Comitato cittadino appositamente costituito.

La Conferenza si svolgerà in due Sezioni parallele: una, sui *mesoni pesanti ed iperoni,* avrà carattere essenzialmente sperimentale, pur includendo una relazione generale sugli aspetti teorici della questione; l'altra, essenzialmente teorica, riguarderà la *teoria quantistica dei campi* e le *interazioni pione-nucleone.* Quest'ultimo argomento sarà preceduto da un'ampia relazione sui risultati sperimentali più recenti.

L'ultimo giorno verrà tenuta una discussione generale a Sezioni riunite preceduta dalle relazioni riassuntive dei lavori svolti nelle due stesse Sezioni.

Inoltre simultaneamente alla Conferenza internazionale avrà luogo a Pisa il XLI Congresso della Società Italiana di Fisica nella sua forma consueta.

Segretario generale di tutta l'organizzazione è il prof. Marcello Conversi, Direttore dell'Istituto di Fisica dell'Università di Pisa (Piazza Torricelli, 2).

Hence (*)

$$(4.5)\qquad \left(\frac{\delta S_i}{S_{il}}\right)_{k_0\to 0} =$$

$$= \frac{\sum_{\boldsymbol{R}}{}' \frac{1}{R} \sum_j \int \frac{\exp[-\alpha k^2]}{\exp[\hbar^2 k^2/2mT]-1} \left\{ \sqrt{\frac{\pi}{2kR}}\, J_{\frac{3}{2}}(kR) - (\boldsymbol{e}\cdot\boldsymbol{\nu})^2 \right| \sqrt{\frac{\pi kR}{2}}\, J_{\frac{5}{2}}(kR) \Big\} \exp[i\boldsymbol{q}\cdot\boldsymbol{R}]\,\mathrm{d}\boldsymbol{q}}{\frac{1}{3}\sum_j \int \frac{k \exp[-\alpha k^2]}{\exp[\hbar^2 k^2/2mT]-1}\,\mathrm{d}\boldsymbol{q}}$$

It is also instructive to compare (4.3) to its short wavelength analogue which, using (2.2), (2.16), (2.10), and the remark at the end of section 2 and neglecting terms of relative order α/v_0^2 may be written

$$(4.6)\qquad \lim_{k_0\to\infty} k_0^2 \langle \delta S_1 \rangle = -\sum_{\boldsymbol{R}}{}' \frac{4(\boldsymbol{\nu}\cdot\boldsymbol{\beta}\boldsymbol{\nu}) - \mathrm{Tr}(\boldsymbol{\beta})}{R^4} =$$

$$= -\frac{v_0}{(2\pi)^3}\frac{m}{M} \sum_{\boldsymbol{R}}{}' \frac{1}{R^4} \sum_j \frac{1}{k^2} \coth \frac{\hbar^2 k^2}{2mT} \left\{ 4(\boldsymbol{\nu}\cdot\boldsymbol{e})^2 - 1 \right\} \cos \boldsymbol{q}\cdot\boldsymbol{R}\,\mathrm{d}\boldsymbol{q}\,.$$

We see that (4.3) is considerably more complicated than (4.6) because of the presence in the integrand of the former expression of an oscillating factor depending on k. While for large R the individual terms in (4.6) decrease as R^{-5}, the convergence of (4.3) is very much poorer, as will be shown later for a specific example. Accordingly, it is convenient to transform (4.5) into τ-space.

Using the relation

$$(4.7)\qquad \int \frac{\exp[i\boldsymbol{s}\cdot\boldsymbol{R}]}{R} \left\{ \left(\frac{\pi}{2kR}\right)^{\frac{1}{2}} J_{\frac{3}{2}}(kR) - \frac{(\boldsymbol{e}\cdot\boldsymbol{R})^2}{R^2} \left(\frac{\pi kR}{2}\right)^{\frac{1}{2}} J_{\frac{5}{2}}(kR) \right\} \mathrm{d}\boldsymbol{R} =$$

$$= 4\pi^2 \frac{(\boldsymbol{e}\cdot\boldsymbol{s})^2}{s^2}\,\delta(s^2 - k^2)\,,$$

where $\boldsymbol{s}$ and $\boldsymbol{e}$ are fixed vectors, we obtain

$$(4.8)\qquad \left(\frac{\delta S_1}{S_{il}}\right) =$$

$$= -1 + \frac{\frac{1}{2\pi}\sum_{\boldsymbol{\tau}} \int \frac{\exp[-\alpha(\boldsymbol{q}+\boldsymbol{\tau})^2]}{\exp[\hbar^2(\boldsymbol{q}+\boldsymbol{\tau})^2/2mT]-1} \sum_j \frac{(\boldsymbol{e}\cdot(\boldsymbol{q}+\boldsymbol{\tau}))^2}{(\boldsymbol{q}+\boldsymbol{\tau})^2}\,\delta\{(\boldsymbol{q}+\boldsymbol{\tau})^2 - k^2(\boldsymbol{q},\boldsymbol{j})\}\,\mathrm{d}\boldsymbol{q}}{\frac{v_0}{(2\pi)^3}\cdot\frac{1}{3}\sum_j \int \frac{k\exp[-\alpha k^2]}{\exp[\hbar^2 k^2/2mT]-1}\,\mathrm{d}\boldsymbol{q}}\,.$$

(*) If δS_1 is considered as an approximation to δS it is convenient to compare it to S_i rather than to S_{il}. The limit $\lim_{k_0\to 0} k_0 S_i$ is usually well represented by the first term of its expansion in powers of the neutron mass ([2]), which is given by (4.4) with the Debye-Waller factor $\exp[-\alpha k^2]$ in the integrand replaced by 1. This approximation to $\delta S/S_i$ is thus obtained from (4.5) by omitting the factor $\exp[-\alpha k^2]$ in the denominator but retaining it in the numerator. In general, the difference between this modified expression and (4.5) is appreciable for light nuclei only.

Eq. (4.8) may also be obtained from the representation in τ-space of $\mathrm{d}S_{c1}/\mathrm{d}\boldsymbol{k}$ given in B ([12]):

$$\lim_{k_0\to 0} k_0 \frac{\mathrm{d}S_{c1}}{\mathrm{d}\boldsymbol{k}} = \frac{1}{2\pi}\frac{m}{M}\frac{\exp[-\alpha k^2]}{\exp[\hbar^2 k^2/2mT]-1}\sum_j \frac{(\boldsymbol{e}(j,\boldsymbol{k})\cdot\boldsymbol{k})^2}{k^2}\,\delta\left\{k^2-\frac{2m}{\hbar}\omega(j,\boldsymbol{k})\right\}. \tag{4.9}$$

Here $\boldsymbol{e}$ and ω are defined as periodic functions of $\boldsymbol{q}$

$$\begin{cases} \boldsymbol{e}(j,\boldsymbol{q}+\boldsymbol{\tau}) = \boldsymbol{e}(j,\boldsymbol{q}) \\ \omega(j,\boldsymbol{q}+\boldsymbol{\tau}) = \omega(j,\boldsymbol{q}). \end{cases} \tag{4.10}$$

The endpoint of the vector $\boldsymbol{k}$ lies on the scattering surface

$$k^2-\frac{2m}{\hbar}\omega(j,\boldsymbol{k}) = 0. \tag{4.11}$$

A general discussion of this surface has been given in B. Denoting the right hand side of (4.9) by $\varphi(\boldsymbol{k})$, we have

$$\lim_{k_0\to 0} k_0 S_{c1} = \int \varphi(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}. \tag{4.12}$$

Dividing $\boldsymbol{k}$-space into cells around the lattice points $\boldsymbol{\tau}$, we may write the integral in (4.12) as a sum over integrals over one cell

$$\lim_{k_0\to 0} k_0 S_{c1} = \sum_{\boldsymbol{\tau}} \int \varphi(\boldsymbol{\tau}+\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}.$$

Using (4.10) and subtracting (4.4), one obtains (4.8).

It should be noted that the individual terms in the sum over τ in (4.8) corresponding to the parts of the scattering surface lying in the cell surrounding $\boldsymbol{\tau}$, have no physical significance except for the very special cases discussed in B, where in certain cells the scattering surface does not touch the cell boundary and is a closed surface around the point $\boldsymbol{\tau}$. Whenever the scattering surface crosses a cell boundary it does so without any discontinuity.

As shown in B, the scattering surface lies within the energy sphere of radius $k_{\max} = (2m\omega_{\max}/\hbar)^{\frac{1}{2}}$ and its general structure depends on the ratio F of the energetically accessible volume in k-space over the volume of a cell,

$$F = \frac{4\pi}{3} k^3_{\max} \cdot \frac{v_0}{(2\pi)^3}. \tag{4.13}$$

The number of terms in the sum over τ in (4.9) is also of order F. In analogy with the results of section **2**, it might be surmised that $(\delta S_1/S_{i1})_{k_0=0}$ decreases for large F with increasing F. In the following we shall see, however, that this surmise is certainly incorrect in the form stated, and that the question of its possible validity in a more restricted sense is without practical interest, since the actual F-values are not large enough for asymptotic considerations to apply.

In order to obtain general orientation on the dependence of $(\delta S_1/S_{i1})_{k_0=0}$ on F, we shall evaluate (4.8) in the Debye approximation. At the same time this simple calculation will illustrate the inadequacy of the Debye-approximation for a quantitative estimate of the long wavelength interference term more clearly than the previous rather elaborate numerical computation using this approximation ([4,9,13-16]).

5. – Debye Approximation at Long Wavelengths.

The Debye approximation assumes the sound velocity c to be independent of j and $\boldsymbol{q}$ so that

$$\omega(j, \boldsymbol{q}) = cq \tag{5.1}$$

and replaces the cell by a sphere of radius

$$q_0 = 2\pi\left(\frac{3}{4\pi v_0}\right)^{\frac{1}{3}} = \left(\frac{6\pi^2}{v_0}\right)^{\frac{1}{3}}. \tag{5.2}$$

Noting that the Debye temperature θ is given by

$$\theta = \hbar\omega_{\max} = \hbar c q_0\,, \tag{5.3}$$

we have

$$F_D = \left(\frac{k_{\max}}{q_0}\right)^3 = \left(\frac{2mc}{\hbar q_0}\right)^3, \tag{5.4}$$

where F_D, as in B, is the value of F in the Debye approximation. It is con-

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([13]) R. Weinstock: *Phys. Rev.*, **65**, 1 (1944).
([14]) J. M. Cassels: *Progr. Nuclear Physics*, **1**, 185 (1950).
([15]) J. M. Cassels: *Proc. Roy. Soc.*, A **208**, 527 (1951).
([16]) R. Latham and J. M. Cassels: *Proc. Phys. Soc.* (*London*), A **65**, 241 (1952).

venient to introduce $f=F_D^{\frac{3}{2}}$. Using (5.3) we may express f in various ways:

$$f = \frac{2mc}{\hbar q_0} \tag{5.5}$$

or

$$f = \frac{2mc^2}{\theta} \tag{5.6}$$

or

$$f = \frac{\theta}{\hbar^2 q_0^2/2m} = \eta \frac{\theta}{E_c}, \tag{5.7}$$

where E_c is the neutron energy at the Bragg cut-off, corresponding to $k_0=\tau_0/2$ and η is a numerical constant, given by

$$\eta = \left(\frac{\tau_0}{2q_0}\right)^2.$$

The values of η for cubic body centered and face centered lattices are, respectively, .818 and .773.

For purposes of calculation f is most conveniently expressed by

$$f = v_0^{\frac{2}{3}} \frac{\theta}{366}, \tag{5.8}$$

where $v_0^{\frac{1}{3}}$ is measured in Ångström units and θ in degrees abs.

A few values of $F_D = f^{\frac{3}{2}}$ have been given in B. A more extensive list of f-values is given at the end of this section.

In order to evaluate (4.8) in the Debye approximation, we introduce the dimensionless variables

$$\boldsymbol{q}' = \frac{\boldsymbol{q}}{q_0}, \qquad \boldsymbol{\tau}' = \frac{\boldsymbol{\tau}}{q_0}, \qquad T' = \frac{T}{\theta}, \qquad \alpha' = \alpha q_0^2. \tag{5.9}$$

From (2.9) and (2.10) we have then

$$\alpha' = \frac{6}{f}\frac{m}{M}\varphi(T'), \tag{5.10}$$

where

$$\varphi(T') = \frac{1}{2}\int_0^1 \coth\frac{q'}{2T'}\, q'\, \mathrm{d}q' = \begin{cases} T' + \dfrac{1}{12T'} & \text{for } T' \gg \dfrac{1}{2\pi}, \\ \dfrac{1}{4} + T'^2 & \text{for } T' \ll 1. \end{cases} \tag{5.11}$$

The equation (4.11) of the scattering surface reduces to

$$(\boldsymbol{\tau}' + \boldsymbol{q}')^2 = f q', \qquad q' \leqslant 1. \tag{5.12}$$

Eq. (5.12) implies

$$\tau' \leqslant f^{\frac{1}{2}} + 1 \tag{5.13}$$

and represents, for all $\boldsymbol{\tau}' \neq 0$ satisfying (5.13), a surface of revolution around the direction of τ', lying within a sphere in $\boldsymbol{k}'$-space with origin at $\boldsymbol{\tau}'$ and radius 1. For $\boldsymbol{\tau}' = 0$ this surface is the sphere

$$q' = f \leqslant 1, \qquad \tau' = 0, \tag{5.14}$$

which exists only for $f \leqslant 1$.

For $\tau' \neq 0$ we may assume without loss of generality

$$\tau' > 1 \tag{5.15}$$

and write (5.12):

$$q'^2 - (f - 2\tau'\mu)q' + \tau'^2 = 0; \qquad q' \leqslant 1, \ \tau' > 1 \tag{5.16}$$

where μ is the cosine of the angle between $\boldsymbol{q}'$ and $\boldsymbol{\tau}'$.

The solution of (5.16) is given by

$$q' = \tfrac{1}{2}[f - 2\tau'\mu - \sqrt{(f - 2\tau'\mu)^2 - 4\tau'^2}] \leqslant 1, \tag{5.17}$$

since the solution with the positive sign of the square root is not compatible with $q' \leqslant 1$, $\tau' > 1$.

Eq. (5.17) shows that q increases monotonically with increasing μ. The lowest value of q', which occurs for $\mu = -1$, is given by

$$q'_{\min} = \frac{f}{2} + \tau' - \sqrt{f\left(\frac{f}{4} + \tau'\right)}. \tag{5.18}$$

This quantity does not exceed 1 if and only if (5.13) is satisfied.

For $\tau' < \sqrt{f} - 1$, (5.17) represents a closed surface around τ', which lies inside the sphere $q' = 1$ without touching it (Figure 1*c*). For $\tau' > \sqrt{f} - 1$ the surface is an open surface reaching the sphere $q' = 1$ for $\mu = ((f-1)/2\tau') - \tau'/2$ (Fig. 1*a*, *b*). Accordingly

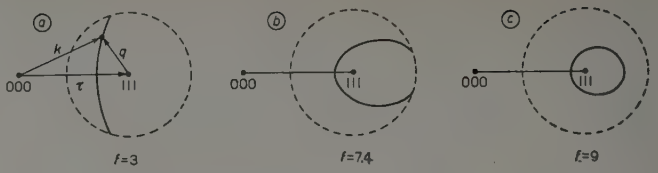

Fig. 1. – Debye approximation to scattering surface in 111 cell for cubic face-centered crystal, for various values of f. The surfaces are surfaces of revolution about the vector $\boldsymbol{\tau}$. The dashed circle represents the sphere $q' = 1$.

$$q'_{\max} = \begin{cases} 1 & \text{for } \tau' \geqslant \sqrt{f} - 1, \\ \dfrac{f}{2} - \tau' - \sqrt{f\left(\dfrac{f}{4} - \tau'\right)} & \text{for } \tau' \leqslant \sqrt{f} - 1. \end{cases} \tag{5.19}$$

Fig. 2 shows the scattering surface resulting from (5.17) for a cubic face centered crystal with $f = 6.3$. It is clear that the description of the scattering surface resulting from the Debye-approximation is quite unrealistic. The shape of the scattering surface depends very sensitively on the frequency-wave vector relationship. From (5.18) and (5.19) it is seen that by far the larger part of the surface lies in a region where q' is not small compared to one and where thus Eq. (5.1) represents the frequency-wave vector relationship very poorly. The continuity of the scattering surface at the cell boundaries is also lost, due to the representation of the cells as overlapping spheres. These defects are likely to manifest themselves quite seriously in the results.

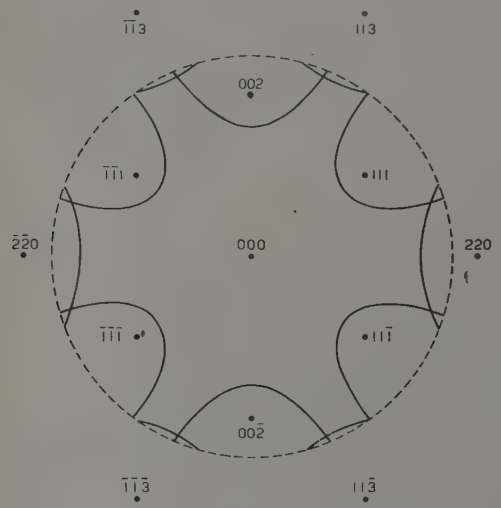

Fig. 2. – Debye approximation to scattering surface for cubic face-centered crystal with $f = 6.3$. The figure represents a cut through the scattering surface in the plane containing the body and face diagonals of the reciprocal cubic body centered lattice. The dashed circle represents the energy sphere $k' = \sqrt{f}$.

Proceeding now to the evaluation of (4.8) in the Debye approximation we obtain, by introducing of (5.1), (5.2), (5.3), (5.5), (5.10),

(5.11) and (5.12)

$$\left(\frac{\delta S_1}{S_{i1}}\right)_{k_0=0} = \tag{5.20}$$

$$2f^{\frac{1}{2}}\psi(f, T')D(f) + f^{-\frac{1}{2}} \sum_{\tau'}^{0<\tau'\leqslant f^{\frac{1}{2}}+1} \frac{1}{2\tau'} \int_{q'_{\min}(\tau', f)}^{q'_{\max}(\tau', f)} \psi(q', T') q' \, dq'$$

$$= -1 + \frac{}{3\int_0^1 \psi(q', T') q'^{\frac{5}{2}} \, dq'},$$

where

$$\psi(q', T') = \frac{\exp\left[-6(m/M)\varphi(T')q'\right]}{\exp\left[q'/T'\right] - 1}, \tag{5.21}$$

with $\varphi(T')$ defined by (5.11) and

$$D(f) = \begin{cases} 1 & \text{for } f < 1, \\ 0 & \text{for } f > 1. \end{cases}$$

The term containing $D(f)$ represents the contribution of the term $\tau' = 0$, which, by (5.14) vanishes for $f > 1$. The limits of the integral in the numerator of (5.20) are given by (5.18) and (5.19).

Eq. (5.20) and (5.21) represent $\delta S_1/S_{i1}$ as a function of f, T' and m/M. We shall discuss this function in the temperature region

$$\frac{1}{2\pi} \ll T' \ll \frac{1}{6}\frac{M}{m}. \tag{5.22}$$

In this region the expression (5.21) for $\psi(q', T')$, is represented by

$$\psi(q', T') = \frac{T'}{q'} \tag{5.21a}$$

so that, for a given lattice type, $\delta S_1/S_{i1}$ becomes a function of f alone.

It has, of course, to be kept in mind that the condition (5.22) cannot be satisfied for light nuclei. Even for heavy nuclei the approximation (5.21a) may, by changing the fraction (5.20) by an amount very small compared to one, change the value of $\delta S_1/S_{i1}$ by an amount of the order of its own size whenever $\delta S_1/S_{i1}$ is very small compared to one. For the study of the general type of f-dependence of the interference term this is, however, without significance and, in view of the much larger uncertainties introduced by the Debye approximation, also without interest for any other purpose.

Introducing (5.21a) into (5.20) and expressing, at the same time, the sum over the vectors $\boldsymbol{\tau}'$ by a sum over their length τ', we obtain

$$\left(\frac{\delta S_1}{S_{i1}}\right)_{k_0=0} = -1 + \frac{5}{3}\sqrt{f}\,D(f) + \frac{5}{6\sqrt{f}}\sum_{\tau'}{}' g(\tau')\chi(\tau', f)\,, \tag{5.23}$$

where $g(\tau')$ is the number of vectors $\boldsymbol{\tau}'$ of length τ' and

$$\chi(\tau', f) = \frac{1}{2\tau'}\left\{q'_{\max}(\tau', f) - q'_{\min}(\tau', f)\right\} = \tag{5.24}$$

$$= \begin{cases} \sqrt{\frac{f}{4\tau'}\left(\frac{f}{4\tau'}+1\right)} - \sqrt{\frac{f}{4\tau'}\left(\frac{f}{4\tau'}-1\right)} - 1 & \text{for } \tau' \leqslant \sqrt{f}-1 \\ \sqrt{\frac{f}{4\tau'}\left(\frac{f}{4\tau'}+1\right)} - \frac{f}{4\tau'} - \frac{1}{2}\left(1-\frac{1}{\tau'}\right) & \text{for } \sqrt{f}-1 \leqslant \tau' \leqslant \sqrt{f}+1 \\ 0 & \text{for } \tau' \geqslant \sqrt{f}+1\,. \end{cases}$$

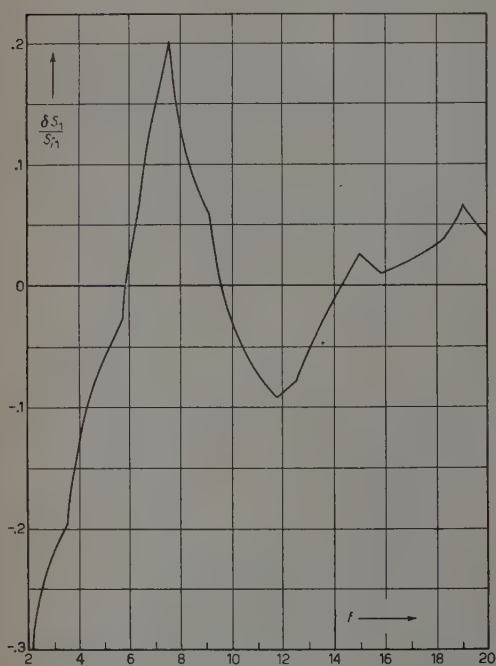

Fig. 3. – Cubic face centered lattice.

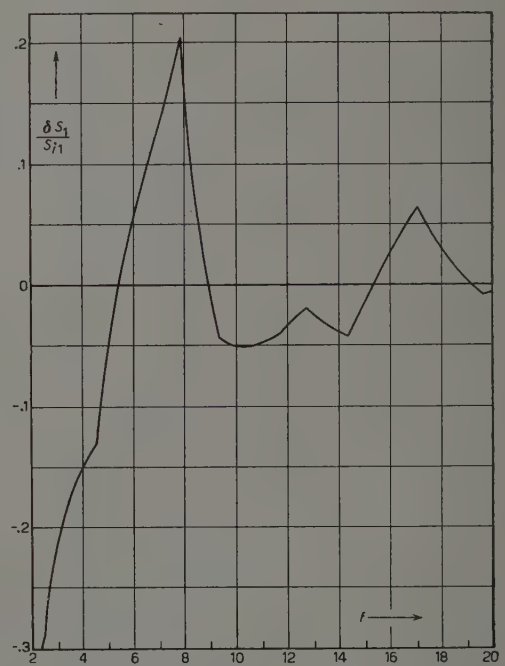

Fig. 4. – Cubic body centered lattice.

The function given by (5.23), which in the following will be denoted by $A(f)$, has an oscillating behaviour. Figs. 3 and 4 show $A(f)$ for the cubic face centered and body centered lattices in the region $2 < f < 20$. The behaviour

in the region $f < 2$ can be directly seen from (5.23) and (5.24). For $f < f_0 = (\tau_0 - 1)^2$, $A(f)$ is given by

$$A(f) = \frac{5}{3}\sqrt{f} - 1\,, \qquad f \leqslant f_0$$

with $f_0 = .57$ for the cubic face centered and $f_0 = .61$ for the cubic body centered lattice. For $f \geqslant f_0$ the sum over $\tau' \neq 0$ comes into play. Accordingly $A(f)$ rises from -1 for $f = 0$ to zero for f in the neighborhood of 0.8 and reaches, for $f = 1$, a value slightly larger than $\frac{2}{3}$, whereupon it drops discontinuously to a value slightly larger than -1. This enormous discontinuity of amount $-\frac{5}{3}$, caused by the sudden disappearance of the scattering surface from the cell around the origin, is entirely due to the unrealistic features of the Debye model. In the correct theory neither the cell nor the scattering surface have spherical shape so that the latter moves gradually from the cell $\tau = 0$ into the surrounding cells as f increases.

For $f > 1$, $A(f)$ rises steeply from a value slightly larger than -1 to zero at $f = 5.8$ (5.4) (*) and reaches a steep maximum at $f = 7.61$ (7.89), where it has a value of about .2, and thereupon exhibits further oscillations.

Table I gives f for a number of crystals. The f values lie all in the region to the left of or at the first maximum shown in the curves. The occurence of small values of $|A|$ is thus not caused by the fact that the number of terms

TABLE I. – *The parameter f.*

Substance	Lattice type	θ in °K	$10^{24} v_0$ in cm^3	f
Pb	f.c.	88	30.0	2.3
Pt	f.c.	225	14.9	3.7
Cu	f.c.	315	11.8	4.5
Ni	f.c.	370	10.9	5.2
Fe–$\alpha\beta$	b.c.	420	11.7	5.9
Al	f.c.	390	16.5	6.9
Cr–α	b.c.	485	11.8	7.0
Mo	b.c.	380	19.9	7.6
Ca	f.c.	230	43.0	7.7

f.c. = cubic face centered
b.c. = cubic body centered
The θ-values are those given by MOTT and JONES [17].

(*) Figures in brackets refer to the body centered lattice.

[17] N. F. MOTT and H. JONES: *The Theory of the Properties of Metals and Alloys* (Oxford, 1936).

in the sum over τ' in (5.20) or (5.23) is large, but is due to the fact that f happens to lie near the point where the interference term changes sign. The difference in the properties of the interference term in the two limiting cases $k_0 \to \infty$ and $k_0 \to 0$ may be readily understood. It is mainly due to two causes. The first lies in the fact that the elastic contribution, which accurately represents the interference term in the former case, plays no role in the latter because of the different wavelength dependence of elastic and inelastic cross-sections. The second cause lies in the different structure of the general parameter F, defined as the ratio of the effective volume in $\boldsymbol{\varkappa}$-space to the volume of a cell. In general, F depends on k_0. For $k_0 = \infty$ the volume is determined by the Debye-Waller factor and accordingly we have, from the considerations of section 2,

$$F(\infty) = \frac{(2\pi)^3}{v_0} \frac{4\pi}{3} \alpha^{-\frac{3}{2}} . \tag{5.25}$$

Introducing again $f = F^{\frac{1}{3}}$ and using (5.2), we may also write (5.25):

$$f(\infty) = \frac{1}{\alpha q_0^2} = \frac{1}{\alpha'} . \tag{5.26}$$

With decreasing k_0 the damping by the Debye-Waller factor becomes incomplete and when k_0^2 becomes of order α^{-1}, f is determined by the energy condition,

$$f(k_0) = \left(\frac{2k_0}{q_0}\right)^2 . \tag{5.27}$$

As soon as k_0^2 becomes comparable to $2m0/\hbar^2$, the energy transfers have to ee included in the energy condition, and for $k_0 = 0$ they alone determine the bffective volume, which yields $f(0)$ as given by (5.6-8).

The oscillatory behaviour of $\langle \delta S \rangle$ is now seen to be a rather general phenomenon, not limited to $k_0 = 0$. It will occur whenever the neighborhood of the energy surfaces in $\boldsymbol{\varkappa}$-space contributes appreciably to $\langle \delta S \rangle$, i.e. whenever f depends on the energy condition. For large k_0, the size of these contributions diminishes with increasing k_0 because of the damping effect of the Debye-Waller-factor. For k_0 large compared to d^{-1} they give rise to oscillations of amplitude small compared to the size of the interference term and this amplitude tends rapidly to zero as k_0 exceeds $\alpha^{-\frac{1}{2}}$. A change in the sign of the interference term as function of f can thus only occur if k_0 is of order d^{-1} or smaller.

To complete this discussion we now compare the representations in $\boldsymbol{R}$-space of $\delta S_1/S_{i1}$ for the two limiting cases $k_0 = \infty$ and $k_0 = 0$ in the Debye approx-

imation. Introducing (4.3*a*) and (5.1) into (4.5) and using the relation

$$\lim_{k_0\to\infty} k_0^2 S_{i1} = \frac{1}{4\alpha}\,, \tag{5.28}$$

which may be obtained from the well-known expression for S_{i1} ([14]), we find, for $T' \gg 1/2\pi$

$$\left(\frac{\langle \delta S_1\rangle}{\langle S_{i1}\rangle}\right)_{k_0=\infty} = -\frac{4}{f^2(\infty)} \sum_r{}' \frac{g(r)}{r^5}\, Si(r)\,, \tag{5.29}$$

where $\boldsymbol{r} = q_0\boldsymbol{R}$, $g(r)$ is the number of vectors $\boldsymbol{r}$ of length r and

$$Si(r) = \int_0^r \frac{\sin x}{x}\,\mathrm{d}x\,.$$

This simple sum has been evaluated in A.

For $k_0 = 0$ we obtain, under the condition (5.22), by introduction of (4.3*a*), (5.1) and (5.21*a*) into (4.5)

$$\left(\frac{\langle \delta S_1\rangle}{\langle S_{i1}\rangle}\right)_{k_0=0} = \left(\frac{\delta S_1}{S_{i1}}\right)_{k_0=0} = A(f) = \tag{5.30}$$

$$= \frac{5}{2} \sum_r{}' \frac{g(r)}{r^2}\, \Re\left\{\exp\left[-\frac{irf}{2}/4\right] \int_{\sqrt{(rf/4)}-1}^{\sqrt{(rf/4)}+1} \exp\left[irx^2\right]\mathrm{d}x - \exp\left[ir\right] \frac{\sin rf^{\frac{1}{2}}}{rf^{\frac{1}{2}}}\right\}.$$

Here f stands for $f(0)$, as usual in this section. The integral in (5.30) may be simply expressed in terms of the Fresnel integrals. For $f^{\frac{1}{2}} \gg 1$ (5.30) reduces, by introduction of the asymptotic representation of the integral

$$\int_y^\infty \exp\left[iu^2\right]\mathrm{d}u = -\frac{\exp\left[iy^2\right]}{2iy}\left\{1 - \frac{1}{4y^2}\right\}, \qquad y \gg 1,$$

to

$$A(f) = -\frac{5}{f} \sum_r{}' \frac{g(r)}{r^2} \frac{\sin r}{r} \cos\left(r\sqrt{f}\right), \qquad f^{\frac{1}{2}} \gg 1. \tag{5.31}$$

The extremely poor convergence of this expression is obvious. It can be shown that it is also due to the absence of damping by the Debye-Waller factor. Without entering into this point we mention only that the approximation (5.21*a*), on which (5.23), (5.30), and (5.31) are based, is fully compatible with arbitrarily large f, since in principle f can be increased indefinitely without

violating the condition (5.22). By a somewhat closer discussion of (5.25) one could probably decide the question whether the amplitudes of the oscillations of $A(f)$ tend to zero as f tends to infinity. Since the curves extend far beyond the range of the actually occurring f-values, this question is however entirely academic.

It would also perhaps be possible to find a mixed representation in τ- and $\boldsymbol{R}$-space for $A(f)$. The use of such a representation was essential in the determination of $I^{(0)}$, carried out in A [cf. (2.3a)]. In the present case it seems hardly worth while, since the evaluation of the representation (5.23) of $A(f)$ in τ-space is quite straightforward. It would be far more important to liberate oneself of the unrealistic assumptions of the Debye-approximation. Our derivation of (5.10) and (5.23) tends to indicate that the long wavelength interference term is far more sensitive to these assumptions than S_i. This is also suggested by the fact that at long wavelength the interference term depends, in the Debye approximation, much stronger on the value of θ adopted than does S_i. In the approximation (5.21a) the latter quantity is seen from (4.4) to be proportional to $\theta^{-\frac{1}{2}}$. The much higher sensitivity of δS to θ is seen directly from the curves, noting that f is proportional to θ. While the fact that the interference term changes sign within the range of actually occurring F-values may well be preserved in the correct theory, it will be prudent to conclude that the Debye approximation provides at best only a qualitative indication of the order of magnitude of the long wavelength interference term.

The situation is very different at short wavelengths. In the wavelength region where the asymptotic representation (2.2) holds, the main part of the interference term ($I^{(0)}$ in (2.3)) is independent of the frequency-wave vector relation, which only affects the small corrections represented by $I^{(1)}$ and $I^{(2)}$. For somewhat longer wavelengths, where $\langle \delta S \rangle$ is approximated by (3.15), this relation enters only, in a rather harmless way, through the mean square displacement α. For discussion of the experimental situation we refer to a forthcoming paper by SQUIRES.

We are indebted to the Istituto Nazionale per le Applicazioni del Calcolo for preparing the graphs given in Figs. 3 and 4. We also wish to thank Dr. G. L. SQUIRES for valuable discussions.

RIASSUNTO

Si discutono in maniera generale gli effetti di interferenza sulla sezione totale di diffusione di neutroni nei cristalli. Si esaminano in dettaglio i casi limiti di lunghezze d'onda piccole e grandi. L'approssimazione di Debye risulta inadeguata per la valutazione quantitativa del termine d'interferenza nel caso di grandi lunghezze d'onda.

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Remarks on the Hydrodynamical Representation and Certain Generalization of Quantum Mechanics.

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(ricevuto il 29 Ottobre 1954)

Recently SCHÖNBERG published two articles on *A Non-Linear Generalization of the Schrödinger and Dirac Equations* (1) and *On the Hydrodynamical Model of the Quantum Mechanics* (2). In the present note we would like to give a few comments on these works, pointing out that part of their contents was already comprised in our previous works (3) concerning the formulation and interpretation of quantum mechanics.

First we shall take up the second article of SCHÖNBERG, which starts from the hydrodynamical model of the Schrödinger equation early suggested by MADELUNG (4). Let us note that we have recently supplemented and developed the Madelung's method in some respects (5) such as, e.g., the introduction of the stress tensor (which we called « quantum stress »), the requirement of a quantum condition for the circulation of flow, or the application to the case of Klein-Gordon equation; and thus certain of SCHÖNBERG's considerations have also been given in our works.

As we there explained (and as SCHÖNBERG does also), the Madelung flow is *quasi-irrotational*; and its generalization to the two-fluid model was required for the representation of a non-relativistic spinning particle. It is, however, also possible to associate a quantum-mechanical motion of the spinning particle with a single rotational flow (6). We shall incidentally touch upon the method: The usual two component wave function (ψ_1, ψ_2) is first represented, for that purpose, by the Eulerian angles θ, φ, χ whose Cayley-Klein parameters are (ψ_1, ψ_2) and the norm R of ψ (7). We thus obtain, besides the equation of continuity, the Euler equation for the velocity function

$$\boldsymbol{v} = \frac{1}{m}\left\{\frac{\hbar}{2}(\nabla\chi + \cos\theta\,\nabla\varphi) - \frac{e}{c}\boldsymbol{A}\right\},$$

derivable from θ, φ, and χ, where $\boldsymbol{A}$ is

(1) M. SCHÖNBERG: *Nuovo Cimento*, **11**, 674 (1954).

(2) M. SCHÖNBERG: *Nuovo Cimento*, **12**, 103 (1954).

(3) T. TAKABAYASI: *Prog. Theor. Phys.*, **8**, 143 (1952); **9**, 187 (1953). These papers are hereafter referred to as I and II respectively.

(4) E. MADELUNG: *Zeits. f. Phys.*, **40**, 332 (1926).

(5) See I and II, especially, I - §§ **3**, **7**, **9**, **11**(*c*), appendices A, B, and II-§§ **5**(*a*), **7** appendices D, F.

(6) See reference (2), § **1**, which mentions unpublished work of BOHM *et al.*.

(7) See, e.g., W. T. PAYNE: *Am. Journ. Phys.*, **20**, 253 (1952); H. A. KRAMERS: *Quantentheorie des Elektrons und der Strahlung* (Akad. Verlagsgesell., Leipzig, 1938), p. 261.

the vector potential for the external electromagnetic field. The Eulerian angles θ, φ, χ in the geometrical representation of the spinor just define the Clebsch parameters for the velocity function which now is a general rotational flow.

In this model, however, the Clebsch potentials θ and φ mean at the same time the direction of spin angular momentum distributed in space, and therefore must be determined uniquely. In fact they can be determined by the equation of motion for them which are to be coupled with the Euler equation. Simultaneous equations of motion and subsidiary conditions for the density, velocity, and spin fields, where the Eulerian angles can perfectly be eliminated, constitute the hydrodynamical picture in an extended sense for the original spinor equation.

We shall now consider Schönberg's method farther. He introduces the operators of mass and current densities ([8]) and stress tensor in the one particle formalism, and thereby suggests, for a quantum-mechanical motion of a single particle, the picture of such « turbulent » flow as its average reduces to the Madelung flow. It is, however, to be remarked that it is no more than a loose analogy, as the rules and meanings of « averages » for the case of quantum operators and for the case of random turbulent variables are essentially different. In fact the so-called turbulence of SCHÖNBERG is not well specified so as to be taken as a definite « model ».

Though he suggests to explain the occurring of the quantum potential in the Madelung flow or in de Broglie's trajectory ensemble ([9]) partly as due to the Reynold's stress, i.e., the turbulent transfer of momentum, it thus remains also a mere analogy. In this connection we would like to note that we have, on the other hand, derived the quantum potential in those methods precisely as the extra momentum flow appearing as the result of projecting onto the coordinate space the Markoff-like process in *phase-space* with transitions in particle momentum, which process represents a quantum-mechanical motion in our still alternative formulation of quantum mechanics ([10]).

SCHÖNBERG also studies what he calls the « quantization of Madelung fluid », emphasizing that it gives rise to a sort of turbulence in the Madelung fluid. In fact, however, his procedure is nothing but the usual method for the quantization of the Schrödinger field, i.e., the second quantization of the Schrödinger particles. Surely such quantized field retains a hydrodynamical aspect, yet one should note that it is not, at least as it is, a pure hydrodynamical field to be defined by density and velocity variables mechanically. The quantization of the Schrödinger field in a hydrodynamical form touched upon in our paper (I-§ **12**), has been attacked by NISHIYAMA ([11]).

It has also been considered occasionally to compare a quantized field to a turbulent flow. Really the quantum fluctuation of a field is analogous to a turbulent fluctuation where the fluid velocity is a random continuous function of space and time. Yet again there should remain essential differences as to their kinematical and statistical properties apart from their dynamics, because the

([8]) Such operators were formerly employed by L. L. LANDAU: *Journ. Phys. USSR*, **5**, 71 (1941).

([9]) The formulation of quantum mechanics, which describes a quantum-mechanical motion of a system by an ensemble of its classical trajectories derivable from a single Hamilton-Jacobi-like function modified by quantum potential, was early suggested by DE BROGLIE, and recently developed by BOHM (*Phys. Rev.*, **84**, 166, 180 (1952)) and also by us (I, II) from different points of view. For one particle problems it is equivalent with the hydrodynamical formulation of Madelung.

([10]) T. TAKABAYASI: *Prog. Theor. Phys.*, **11**, 341 (1954).

([11]) T. NISHIYAMA: *Prog. Theor. Phys.*, **8**, 655 (1952).

usual turbulence is in general to be specified by a probability functional $W[\boldsymbol{v}(x)]$ for the random velocity field $\boldsymbol{v}(x)$, while a state of a quantum field is given by the probability *amplitude* functional such as $\Psi[Q(x)]$, leading to the interference of probabilities.

We shall next turn to SCHÖNBERG's first paper. Since the Schrödinger equation corresponds to the Madelung flow that satisfies the quasi-irrotationality condition, we may obtain, omitting this subsidiary condition, a sort of non-linear generalization of the Schrödinger equation so as to correspond to including a general rotational flow. Such attempt, to which SCHÖNBERG's first paper is devoted, was already made ([12]) in I-§ **13**, together with the representation in terms of Clebsch variables worked out in II-appendix E. In our case further the quantization of that generalized Schrödinger field ([13]) was considered.

The application of the above procedure to relativistic cases was attempted in different ways: SCHÖNBERG generalizes the Dirac equation by means of formal introductions of Clebsch parameters independently of a hydrodynamical model, while we have generalized the Klein-Gordon equation keeping to the hydrodynamical analogy and represented it in *generalized* Clebsch variables (II-§ **5** (*b*)).

In conclusion we would like to note that recently we have also analysed and criticized various other attempts ([14]) of « non-linear generalization of the Schrödinger equation » ([15]).

([12]) This point was more lately noticed by SCHÖNBERG also (*Nuovo Cimento*, **12**, 300 (1954)).

([13]) This idea was commented on by BOHM (*Prog. Theor. Phys.*, **9**, 273 (1953): see the last paragraph). Also cf. F. A. KAEMPFFER: *Can. Journ. Phys.*, **32**, 259 (1954).

([14]) See, e.g., L. JÁNOSSY: *Ann. d. Phys.*, (6) **11**, 323 (1953).

([15]) T. TAKABAYASI: *Soryuɘiron-kenkyu*, **7**, No. 1, 21 (1954) (in Japanese); also to be published elsewhere.

Sul moto di un elettrone investito da un impulso elettromagnetico istantaneo.

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(ricevuto il 23 Novembre 1954)

Recentemente è stata proposta da Caldirola [1] una equazione pel moto classico dell'elettrone, atta a superare alcune caratteristiche difficoltà delle teorie precedenti. Si tratta di una equazione, relativisticamente invariante, alle differenze finite che si fonda sull'introduzione nella teoria di una lunghezza elementare [2]. Essa si riduce, nell'approssimazione dei moti lentamente variabili, all'equazione differenziale del terzo ordine di Dirac [3].

Ci proponiamo di studiare, partendo dall'equazione di Caldirola, il moto di un elettrone investito da un impulso elettromagnetico istantaneo.

Tale problema è stato trattato, colla corrispondente equazione di Dirac, da Dirac stesso [3] nell'approssimazione non relativistica e da Ashauer [4] nel caso relativistico. I risultati ottenuti da questi Autori mettono in luce un comportamento alquanto singolare dell'elettrone. Supposto infatti che esso sia inizialmente a riposo, incomincia ad accelerare prima ancora di essere raggiunto dall'impulso elettromagnetico per poi assumere una velocità costante dopo che l'impulso è passato oltre. L'interpretazione di tale risultato richiede o l'introduzione di un modello di elettrone piuttosto complicato e non privo di contraddizioni o l'abbandono dei tradizionali concetti su cui è fondata la causalità della fisica classica.

Schematizzato l'impulso elettromagnetico che raggiunge l'elettrone all'istante di tempo proprio τ^* nel seguente modo:

$$(1)\qquad \begin{cases} E_x(\tau) = \begin{cases} A_0 & \text{per } \tau = \tau^* = r\tau_0 \\ 0 & \text{per } \tau \neq \tau^* \end{cases} \qquad E_y = E_z = 0\,, \\ H_x = H_z = 0\,, \qquad H_y(\tau) = \begin{cases} A_0 & \text{per } \tau = \tau^* = r\tau_0 \\ 0 & \text{per } \tau \neq \tau^* \end{cases} \end{cases}$$

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[1] P. Caldirola: *Nuovo Cimento*, **10**, 1747 (1953).
[2] P. Caldirola e F. Duimio: *Nuovo Cimento*, **12**, 699 (1954).
[3] P. A. M. Dirac: *Proc. Roy. Soc.*, A **167**, 148 (1938).
[4] S. Ashauer: *Proc. Camb. Phil. Soc.*, **45**, 463 (1949).

FONDAZIONE FRANCESCO SOMAINI

PRESSO IL TEMPIO VOLTIANO, A COMO

BANDI DI CONCORSI

AL PREMIO E ALLA BORSA PER IL 1955

Con lo scopo di premiare e incoraggiare nel nome di ALESSANDRO VOLTA gli studi di Fisica in Italia, la « Fondazione Francesco Somaini » presso il Tempio Voltiano a Como, indice i seguenti Concorsi:

A) **Concorso al " Premio Triennale per la Fisica Francesco Somaini " per il 1955** di L. 1 500 000 (un milione e cinquecentomila) nette, da assegnarsi al concorrente che, fra quelli che la Commissione Giudicatrice giudicherà in senso assoluto meritevoli del premio per i risultati conseguiti nello studio della Fisica durante il Triennio 1° Luglio 1952, 30 Giugno 1955, sia, a parere della Commissione stessa, il più meritevole.

B) **Concorso alla " Borsa Francesco Somaini per lo studio della Fisica " per il 1955** di L. 750 000 (settecentocinquantamila) nette, da assegnarsi al concorrente che, fra quelli che la Commissione Giudicatrice giudicherà in senso assoluto meritevoli della Borsa, verrà dalla Commissione stessa giudicato il più meritevole, sia per titoli, preparazione scientifica, lavori già svolti e risultati già conseguiti nella Fisica, sia anche per il vantaggio che gli studi, per i quali è richiesta la Borsa, possono portare allo sviluppo della Fisica in Italia.

1. – Ad entrambi i Concorsi possono prendere parte singolarmente i cittadini d'ambo i sessi italiani e svizzeri del Canton Ticino purchè di stirpe italiana. Sono esclusi dal Concorso i membri della Commissione Amministratrice e della Commissione Scientifica della « Fondazione Francesco Somaini ».

2. – Le norme particolareggiate dei singoli Concorsi verranno pubblicate in apposito volantino che potrà essere richiesto dagli interessati alla Segreteria della Fondazione presso il Tempio Voltiano a Como.

3. – La domanda, i documenti, i lavori, ecc., presentati dai singoli concorrenti dovranno pervenire, tra *il 1° Gennaio e le ore 12 del 1° Luglio 1955*, alla Commissione Amministratrice della « Fondazione Francesco Somaini » a Como presso il Tempio Voltiano.

La procedura dei suddetti Concorsi è regolata secondo lo Statuto della Fondazione, il quale è ostensibile a Como presso il Tempio Voltiano ed è depositato presso il Notaio Dr. Raoul Luzzani di Como.

Como, dal Tempio Voltiano — *Il giorno 26 Settembre 1952*

Il Segretario
Conservatore del Tempio
CESARE MORLACCHI

Il Presidente
Sindaco di Como
GIUSEPPE TERRAGNI

l'equazione relativistica di Caldirola

$$-\frac{m_0}{\tau_0}\{u_\alpha(n\tau_0-\tau_0)+u_\alpha(n\tau_0)u_\beta(n\tau_0)u_\beta(n\tau_0-\tau_0)\}=eF_{\alpha\beta}u_\beta(n\tau_0)\,,$$

diventa in questo caso:

$$(2)\qquad \begin{cases} u_\alpha(n\tau_0-\tau_0)+u_\alpha(n\tau_0)u_\beta(n\tau_0)u_\beta(n\tau_0-\tau_0)=0 & \text{per } n\neq r \\ u_\alpha(r\tau_0-\tau_0)+u_\alpha(r\tau_0)\,u_\beta(r\tau_0)u_\beta(r\tau_0-\tau_0)=-\lambda f_\alpha(r\tau_0) & \text{per } n=r \end{cases}$$

con

$$\lambda=\frac{e\tau_0 A_0}{m_0}\,,$$

$$f_1(r\tau_0)=-iu_4(r\tau_0)-u_3(r\tau_0)\,;\quad f_2(r\tau_0)=0\,;\quad f_3(r\tau_0)=f_4(r\tau_0)=u_1(r\tau_0)\,.$$

Vogliamo notare esplicitamente come la schematizzazione (1) dell'impulso differisca da quella cui si ricorre nel corrispondente problema trattato con l'equazione di Dirac. In questo caso infatti, dovendo essere la velocità una funzione continua per tutti i valori della variabile τ, si suole porre:

$$E_x(\tau)=H_y(\tau)=K\delta(\tau-\tau^*)\,.$$

Le soluzioni delle (2), soddisfacenti alle condizioni iniziali ($u_1^0=u_2^0=u_3^0=0$, $u_4^0=i$), si trovano facilmente:

$$u_1(n\tau_0)=u_2(n\tau_0)=u_3(n\tau_0)=0\,,\qquad u_4(n\tau_0)=i\,,\qquad \text{per } n<r$$

$$u_1(n\tau_0)=\frac{\sqrt{1+4\lambda^2}-1}{2\lambda}\,;\quad u_2(n\tau_0)=0\,;\quad u_3(n\tau_0)=\frac{\sqrt{1+4\lambda^2}-1}{2}\sqrt{\frac{\sqrt{1+4\lambda^2}-1}{2}}\,;$$

$$u_4(n\tau_0)=i\sqrt{\frac{1+\sqrt{1+4\lambda^2}}{2}}\,,\qquad \text{per } n\geq r$$

Passando all'ordinario spazio tridimensionale si osserva quindi che l'elettrone resta in quiete fino a quando l'impulso lo colpisce, mentre dopo si muove nel piano $y=0$ con velocità uniforme di modulo

$$(3)\qquad v=\frac{\sqrt{1+4\lambda^2}-1}{2\lambda}\,,$$

lungo una retta di coseni direttori

$$(4)\qquad \frac{v_x}{v}=\frac{1}{\lambda}\sqrt{\frac{\sqrt{1+4\lambda^2}-1}{2}}\,;\quad \frac{v_z}{v}=\frac{1}{2\lambda}(\sqrt{1+4\lambda^2}-1)\,.$$

Dalla (3) si vede che per $\lambda \ll 1$ (approssimazione non relativistica) è

$$v = \frac{e\tau_0 A_0}{m_0} .$$

Si noti che nel corrispondente problema trattato prendendo l'equazione di Dirac, si trova nella stessa approssimazione come valore della velocità dell'elettrone dopo essere stato investito dall'impulso:

$$v = \frac{eK}{m_0} ,$$

onde confrontando col nostro risultato si vede che le velocità asintotiche nelle due teorie coincidono (nell'approssimazione non relativistica) pur di assumere

$$A_0\tau_0 = K .$$

Tale velocità asintotica è poi la stessa che acquisterebbe l'elettrone non irraggiante (pel quale varrebbe l'ordinaria equazione di Newton) per effetto di un campo elettrico costante di intensità A_0 che agisse per un intervallo di tempo τ_0.

Dalla (4) si vede poi che per $\lambda \to \infty$ si ha:

$$\frac{v_x}{v} \to 0 , \qquad \frac{v_z}{v} \to 1 , \qquad v \to 1 ;$$

cioè quando l'impulso è grandissimo in intensità l'elettrone tende a muoversi nella stessa direzione di propagazione dell'impulso con velocità che tende a quella della luce al crescere di tale intensità. Un comportamento analogo si aveva per l'equazione relativistica di Dirac.

Possiamo quindi concludere che il comportamento dell'elettrone, dopo che esso è stato investito dall'impulso elettromagnetico, è sostanzialmente lo stesso nella teoria di Caldirola che in quella di Dirac.

Un comportamento diverso si ha invece pel moto negli istanti precedenti all'arrivo dell'impulso: le soluzioni dell'equazione alle differenze finite (2) non mostrano quel comportamento strano che caratterizzava le soluzioni delle corrispondenti equazioni differenziali del terzo ordine di Dirac.

High Energy Electron Scattering by Polarized Nuclei.

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(ricevuto il 29 Novembre 1954)

Experiments on the elastic scattering of high energy electrons by nuclei have been recently carried out (1), giving relevant information about the nuclear charge distribution (2). Until now only unpolarized target nuclei have been used, so only the radial charge distribution could be studied. It has been possible, recently, to align nuclei of paramagnetic substances to a high degree (3). Therefore the measurement of the azimuthal dependence of electron-nucleon cross-section due to the asymmetry in the nuclear charge distribution has come within the range of the actual experimental possibilities.

The calculation of the electron scattering cross-section by a given asymmetric charge distribution involves the solution of the following Dirac equation:

$$[(\boldsymbol{\alpha p})\, c + \beta mc^2 + V(\boldsymbol{r})]\psi = E\psi \,. \tag{1}$$

If we take into account only quadrupole effects in the nuclear charge distribution, $V(\boldsymbol{r})$ will be written as

$$V(\boldsymbol{r}) = V_0(r) + \eta V_2(r) T_0^2(I_r) \,, \tag{2}$$

where η is the nuclear eccentricity, I_r is the r component of the nuclear spin and T_0^2 is the tensor operator (4) given by:

$$T_0^2(I_r) = \frac{3I_r^2 - I(I+1)}{I(2I-1)} \,. \tag{3}$$

The solution of eq. (1) is a difficult task since the potential is not spherically symmetrical, so of course, the different values of l are mixed together.

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(1) R. Hofstadter, B. Hahn, A. W. Knudsen and J. A. Mc Intyre: *Phys. Rev.*, **95**, 512 (1954).

(2) D. G. Ravenhall and D. R. Yennie: *Phys. Rev.*, **96**, 239 (1954); D. R. Yennie, D. G. Ravenhall and R. N. Wilson: *Phys. Rev.*, **95**, 500 (1954); S. Brenner, G. E. Brown and L. B. R. Elton: *Phil. Mag.*, **45**, 524 (1954).

(3) A. W. Overhauser: *Phys. Rev.*, **8**, 689 (1953); **92**, 411 (1953); P. Brovetto and G. Cini: *Nuovo Cimento*, **11**, 618 (1954), quoted as B.C.; P. Brovetto and S. Ferroni: *Nuovo Cimento*, **12**, 90 (1954); M. E. Rose: *Phys. Rev.*, **75**, 213 (1949); C. J. Gorter: *Physica*, **14**, 504 (1948).

(4) G. Racah: *Phys. Rev.*, **62**, 438 (1942); A. Simon: *Phys. Rev.*, **92**, 1050 (1953).

In this paper we shall give a preliminary investigation of the main features of the phenomenon that can be deduced without solving explicitely eq. (1).

Using general requirements of symmetry and neglecting higher order effects in η, we get for the scattering matrix M the following expression:

$$M = M_0(\theta)\left[1 + \eta f_1(\theta) T_0^2(I_\alpha) + \eta f_2(\theta) T_0^2(I_\beta)\right], \tag{4}$$

where $M_0(\theta)$, $f_1(\theta)$, $f_2(\theta)$ are complex functions of θ and of the product kR [5] and I_α and I_β are the projections of I on the directions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of $\boldsymbol{p}_1 - \boldsymbol{p}_2$ and $\boldsymbol{p}_1 + \boldsymbol{p}_2$ respectively ($\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ represent of course the initial and final electron momenta).

Since the target nuclei are not in a pure spin state, we shall use the density matrix ϱ [6] defined by

$$\varrho = \sum P_m \rangle_{m\, m}\langle , \tag{5}$$

ϱ will be generally of the form

$$\varrho = \exp\left[-I_h \mu\right], \tag{6}$$

where h is the direction of the magnetic field.

We define the polarization P and the alignement coefficient γ as

$$P = \frac{\operatorname{Tr} I_h \varrho}{I \operatorname{Tr} \varrho}, \tag{7}$$

$$\gamma = \frac{\operatorname{Tr} \varrho T_0^2(I_h)}{\operatorname{Tr} \varrho}, \tag{8}$$

from (6), (7) and (8) it follows that

$$\gamma = \left[1 - \frac{3}{2} \frac{P \operatorname{cotgh} \mu/2}{I+1}\right] \frac{I+1}{I-\frac{1}{2}}. \tag{9}$$

The general formula for the angular distribution σ is

$$\sigma = \frac{\operatorname{Tr} \varrho M^+ M}{\operatorname{Tr} \varrho}, \tag{10}$$

substitution of (5) into (10) gives

$$\sigma = \operatorname{Tr} \varrho \left| M_0(\theta) \right|^2 \left[1 + 2\eta(\varphi_1(\theta) T_0^2(I_\alpha) + \varphi_2(\theta) T_0^2(I_\beta) + \right. \\ \left. + \eta^2 \left| f_1(\theta) T_0^2(I_\alpha) + f_2(\theta) T_0^2(I_\beta) \right|^2\right], \tag{11}$$

where $\varphi_1(\theta)$ and $\varphi_2(\theta)$ are the real parts of $f_1(\theta)$ and $f_2(\theta)$.

The term in η represents the main contribution to our effect. We shall neglect

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[5] This happens because at high energy the term βmc^2 can be neglected.

[6] J. VON NEUMAN: *Die Mathematische Grundlagen der Quantenmechanik* (Leipzig, 1932), p. 173; L. WOLFENSTEIN and J. ASHKIN: *Phys. Rev.*, **85**, 947 (1952); A. SIMON, M. E. ROSE and J. M. JAUCH: *Phys. Rev.*, **84**, 1155 (1951).

the term in η^2: this term has already been considered by SCHIFF (7) in the case of no polarization ($\varrho=1$). Using (8) and recalling some properties of the tensor operators we get easily:

$$\sigma = |M_0(\theta)|^2\left[1 + 2\eta\gamma[\varphi_1(\theta)P_2(\boldsymbol{\alpha h}) + \varphi_2(\theta)P_2(\boldsymbol{\beta h})]\right]. \tag{12}$$

Relation (12) shows that the best experimental conditions for determining φ_1 and φ_2 are to be found when both angles $(\boldsymbol{p}, \boldsymbol{h})$ and θ are of the order of 90°.

As we have already seen φ_1 and φ_2, the form of the charge distribution being the same, depend only on kR and Z. Therefore scattering experiments on different isotopes may be rather interesting. It is well known that the addition of one or more neutrons to a nucleus can change considerably its quadrupole moment; the experiment proposed here may indicate if the charge distribution responsible for these quadrupole moments is also changed.

We proceed now to evaluate the order of magnitude to be expected for this azimuthal effect. As an example let us consider the case of ^{181}Ta polarized be means of the Overhauser effect. The reason of our choice is that ^{181}Ta has a rather big eccentricity $\eta=0.14$ and its spin 7/2 allows to obtain high polarization for not too small values of the temperature. The coefficient μ for Overhauser effect is given by

$$\mu = \frac{\gamma_n + s\gamma_e}{RT} H \tag{13}$$

the symbols having the same meaning as in B.C.. With $s=0.9$, $H=10\,700$ gauss and $T=2$ °K, $\mu=0.66$, $P=0.70$ and $\gamma=0.40$ (8).

If we consider 100 MeV electrons scattered at 90°, a rough evaluation based on the Born approximation gives $\varphi_1(\theta) \sim 4$, $\varphi_2(\theta)=0$; substituting in (12) we get an azimuthal effect of the order of 40%.

This numerical result is not to be taken very seriously because the Born approximation is not reliable in this type of problems. We believe that the correct results are somewhat smaller than the ones obtained by the Born approximation, altough the azimuthal effect is likely to remain appreciable. We envisage, in the near future, to give a treatment of the scattering problem using a better type of approximation.

We are grateful to Profs. A. GAMBA and R. MALVANO for very useful discussion and to Prof. G. WATAGHIN for his kind interest in this work.

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(7) L. SCHIFF: *Phys. Rev.*, **92**, 988 (1953).
(8) See formula (16) of B.C.

Schermi di rinforzo per autoradiografia.

M. Ageno

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(ricevuto il 9 Dicembre 1954)

Una esigenza ormai vivamente sentita in molti campi della ricerca sia pura sia applicata è quella di ridurre al minimo il tempo necessario per la raccolta dei dati sperimentali e per la elaborazione di essi.

Credo perciò non inutile segnalare un semplice artificio suscettibile in alcuni casi di abbreviare notevolmente il tempo necessario per l'esecuzione di una autoradiografia.

Questa tecnica fotografica viene, com'è ben noto, correntemente impiegata nella ricerca bio-chimica in unione alle tecniche cromatografiche per lo studio, ad esempio, di reazioni di sintesi partendo da sostanze marcate con radioisotopi. I fogli di carta, portanti i cromatogrammi mono- o bi-dimensionali, vengono tenuti per un certo tempo a contatto con una pellicola o lastra fotografica, in modo che l'auto-radiografia riveli la posizione e la intensità delle macchie contenenti il tracciante. Il tempo di esposizione deve spesso essere assai lungo, dell'ordine del mese o più ancora, tanto che ci si avvicina all'ordine di grandezza del tempo di affievolimento spontaneo dell'immagine latente. Talora, anche, il periodo di dimezzamento del radioisotopo, relativamente breve, impedisce l'impiego di esposizioni prolungate e costringe (quando possibile) ad aumentare proporzionalmente la dose iniziale del tracciante.

È tuttavia da osservare che non sempre il percorso degli elettroni emessi dal radioisotopo, posto a contatto con la lastra fotografica si esaurisce completamente nella gelatina sensibile. Si offre allora la possibilità di intensificare, a parità di dose, l'azione fotografica del tracciante, in modo simile a quello in uso da tempo per la röntgen-grafia.

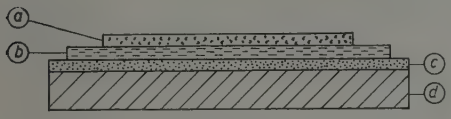
In quest'ultimo caso, la pellicola fotografica è compressa tra due schermi fluorescenti, che rivolgono ad essa la pagina sensibile. I due schermi vengono attraversati praticamente senza attenuazione dal fascio di raggi X incidente e la luce di fluorescenza che da essi si sprigiona contribuisce fortemente ad impressionare la gelatina sensibile.

Nel nostro caso, l'attenuazione subita dalla radiazione elettronica è in genere già elevata dopo attraversamento della gelatina, totale o quasi totale dopo attraversamento del supporto di essa. Non è quindi possibile di solito l'impiego, *sic et simpliciter*, degli schermi di rinforzo per raggi X.

La prima idea che si presenta alla

mente è allora quella di sostituire al celluloide della pellicola fotografica un foglio di una plastica fluorescente agli elettroni. Ciò per altro richiede la fabbricazione di pellicole speciali, cosa non agevole da realizzare in laboratorio.

Una seconda soluzione, certamente assai meno conveniente della prima ma che pur si è dimostrata conveniente almeno in alcuni casi, consiste (fig. 1) nell'interporre tra la sorgente di elettroni



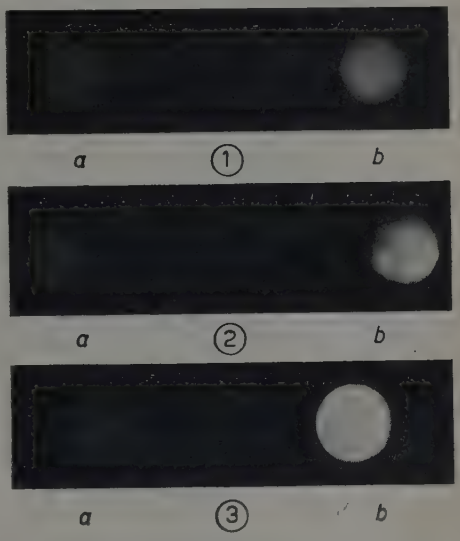
a) Preparato; *b*) Schermo fluorescente; *c*) Gelatina sensibile; *d*) Supporto.

Fig. 1. – Posizione dello schermo di rinforzo.

e la gelatina della pellicola fotografica uno schermo fluorescente di spessore opportuno.

L'attenuazione dell'azione fotografica per effetto dell'assorbimento nello schermo può infatti essere, in condizioni favorevoli, più che compensata dal rinforzo dovuto alla luce di fluorescenza. Le condizioni in cui questo avviene vanno naturalmente scelte di volta in volta, in relazione all'energia degli elettroni emessi dal tracciante e alla sensibilità e al tipo dell'emulsione usata. Non è escluso che possa talora convenire rinunciare addirittura all'azione fotografica diretta degli elettroni e usare questi solo come eccitatori della luce di fluorescenza nello schermo.

A titolo di esempio e per mostrare il vantaggio che si può trarre da questo metodo, riportiamo nella fig. 2 i risultati di alcune prove eseguite con lastre Cappelli-Ferrania ortocromatiche, 16/10 din., che sono usate correntemente nel nostro Laboratorio per la microscopia e la diffrazione elettronica, data la loro buona sensibilità agli elettroni. Come sorgente di elettroni si è usato un preparatino di ^{90}Sr, che è stato deposto due volte successivamente e per tempi uguali sulla stessa lastra, in punti diversi: la prima volta (*a*), direttamente, la seconda (*b*), con interposizione di uno schermo fluorescente.



1) Plastica al tetrafenilbutadiene. 2) Schermo al ZnS. 3) Schermo al $CaWO_4$.

Fig. 2. – Rinforzi ottenuti con vari schermi.

Le tre prove riportate nella figura sono state eseguite con i seguenti schermi:

1) Foglio di plastica fluorescente, al tetrafenilbutadiene, dello spessore di mm 1.

2) Schermo fluorescente al solfuro di zinco su supporto in vetro trasparente.

3) Schermo fluorescente al tungstato di calcio su supporto in cartoncino leggero.

Per meglio valutare il vantaggio offerto dallo schermo di rinforzo si sono misurate le densità ottiche al centro delle macchie della fig. 2.

I risultati sono contenuti nella tabella I.

Tabella I.

Rinforzo	Densità ottica	
	a	*b*
Plastica al tetrafenil-butadiene	0,32	0,59
Schermo al solfuro di zinco	0,26	0,74
Schermo al tungstato di calcio	0,23	1,37

Com'è ovvio, tali dati hanno un valore esclusivamente indicativo, venendo a dipendere in modo assai sensibile dal tipo e spessore del supporto dello schermo e dallo spessore dello strato fluorescente.

La fig. 2 mostra anche come la perdita di potere risolutivo certamente causata dalla interposizione dello schermo fluorescente tra preparato e pellicola, si mantenga entro limiti del tutto accettabili.

Sull'irraggiamento dell'elettrone nell'elettrodinamica classica.

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(ricevuto il 10 Dicembre 1954)

Recentemente abbiamo stabilito [1] per il moto dell'elettrone irraggiante « classico » la seguente equazione relativistica:

$$(1) \qquad -\frac{m_0}{\tau_0}\left[u_\alpha(\tau-\tau_0)+\frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}u_\beta(\tau-\tau_0)\right]=\frac{e}{c}F^{(\mathrm{ext})}_{\alpha\beta}(\tau)u_\beta(\tau)\,,$$

in cui τ_0 e m_0 risultano fra loro legati dalla relazione:

$$m_0=\frac{4e^2}{3\tau_0 c^3}\,.$$

Abbiamo anche mostrato come la (1) possa essere dedotta dalle equazioni maxwelliane del campo e.m.. Da questo punto di vista il campo proprio associato all'elettrone in moto risulta dato da:

$$F^{(\mathrm{self})}_{\alpha\beta}(\tau)=\frac{m_0}{e\tau_0 c}\left[u_\alpha(\tau)u_\beta(\tau-\tau_0)-u_\alpha(\tau-\tau_0)u_\beta(\tau)\right];$$

con che la (1) si scrive:

$$\frac{e}{c}\left[F^{(\mathrm{self})}_{\alpha\beta}+F^{(\mathrm{ext})}_{\alpha\beta}\right]u_\beta=0\,.$$

Osserviamo ora come l'espressione della forza propria:

$$K^{(\mathrm{self})}_\alpha=\frac{e}{c}F^{(\mathrm{self})}_{\alpha\beta}u_\beta\,,$$

[1] P. Caldirola: *Nuovo Cimento*, **10**, 1747 (1953); P. Caldirola e F. Duimio: *Nuovo Cimento*, **12**, 699 (1954).

possa essere scomposta nella somma di due termini:

$$K_\alpha^{(\mathrm{self})} = K_\alpha^{(\mathrm{reaz})} + K_\alpha^{(\mathrm{diss})} ,$$

ove si ponga:

$$K_\alpha^{(\mathrm{reaz})}(\tau) = -\frac{m_0}{2\tau_0}\left\{ u_\alpha(\tau+\tau_0) - u_\alpha(\tau-\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}\left[u_\beta(\tau+\tau_0) - u_\beta(\tau-\tau_0)\right]\right\},$$

$$K_\alpha^{(\mathrm{diss})}(\tau) = \frac{m_0}{2\tau_0}\left\{ u_\alpha(\tau+\tau_0) + u_\alpha(\tau-\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}\left[u_\beta(\tau+\tau_0) + u_\beta(\tau-\tau_0)\right]\right\}.$$

Il termine $K_\alpha^{(\mathrm{reaz})}$ che, come si verifica immediatamente sviluppando secondo le potenze di τ_0, contiene termini contenenti tutte le derivate di ordine dispari della velocità $u_\lambda(\tau)$, ha carattere *conservativo*. Esso generalizza la forza d'inerzia $-m_0(\mathrm{d}u_\alpha/\mathrm{d}\tau)$ dell'equazione relativistica della particella non irraggiante: le correzioni apportate a questo termine (*correzioni radiative*) sono dovute alla reazione del campo proprio dell'elettrone. $K_\alpha^{(\mathrm{reaz})}$ può pensarsi associato al campo:

$$F_{\alpha\beta}^{(\mathrm{reaz})}(\tau) = \frac{m_0}{2e\tau_0 c}\left\{u_\alpha(\tau)\left[u_\beta(\tau-\tau_0) - u_\beta(\tau+\tau_0)\right] - u_\beta(\tau)\left[u_\alpha(\tau-\tau_0) - u_\alpha(\tau+\tau_0)\right]\right\},$$

secondo la relazione

$$K_\alpha^{(\mathrm{reaz})} = \frac{e}{c} F_{\alpha\beta}^{(\mathrm{reaz})} u_\beta .$$

Il termine $K_\alpha^{(\mathrm{diss})}$ contiene invece le derivate di ordine pari della velocità ed ha pertanto carattere *dissipativo*: esso è il responsabile dell'energia dissipata sotto forma di radiazione emessa dall'elettrone. Tale termine può pensarsi associato al campo

$$F_{\alpha\beta}^{(\mathrm{diss})}(\tau) = \frac{m_0}{2e\tau_0 c}\left\{u_\alpha(\tau)\left[u_\beta(\tau-\tau_0) + u_\beta(\tau+\tau_0)\right] - u_\beta(\tau)\left[u_\alpha(\tau-\tau_0) + u_\alpha(\tau+\tau_0)\right]\right\}.$$

Le componenti esplicite del vettore elettrico e di quello magnetico dei vari campi si scrivono:

$$\boldsymbol{E}^{(\mathrm{self})}(\tau) = \frac{m_0}{e\tau_0}\frac{1}{\sqrt{(1-\beta^2)(1-\beta_-^2)}}\left\{\boldsymbol{v}(\tau-\tau_0) - \boldsymbol{v}(\tau)\right\},$$

$$\boldsymbol{E}^{(\mathrm{reaz})}(\tau) = \frac{m_0}{e\tau_0}\frac{1}{\sqrt{1-\beta^2}}\left\{\frac{\boldsymbol{v}(\tau-\tau_0)}{\sqrt{1-\beta_-^2}} - \frac{\boldsymbol{v}(\tau+\tau_0)}{\sqrt{1-\beta_+^2}} - \boldsymbol{v}(\tau)\left[\frac{1}{\sqrt{1-\beta_-^2}} - \frac{1}{\sqrt{1-\beta_+^2}}\right]\right\},$$

$$\boldsymbol{E}^{(\mathrm{diss})}(\tau) = \frac{m_0}{2e\tau_0}\frac{1}{\sqrt{1-\beta^2}}\left\{\frac{\boldsymbol{v}(\tau-\tau_0)}{\sqrt{1-\beta_-^2}} + \frac{\boldsymbol{v}(\tau+\tau_0)}{\sqrt{1-\beta_+^2}} - \boldsymbol{v}(\tau)\left[\frac{1}{\sqrt{1-\beta_-^2}} + \frac{1}{\sqrt{1-\beta_+^2}}\right]\right\},$$

$$\boldsymbol{H}^{(\mathrm{self})}(\tau) = \frac{\boldsymbol{v}(\tau)}{c} \wedge \boldsymbol{E}^{(\mathrm{self})}(\tau), \qquad \beta = \frac{v(\tau)}{c},$$

$$\boldsymbol{H}^{(\mathrm{reaz})}(\tau) = \frac{\boldsymbol{v}(\tau)}{c} \wedge \boldsymbol{E}^{(\mathrm{reaz})}(\tau), \qquad \beta_- = \frac{v(\tau-\tau_0)}{c},$$

$$\boldsymbol{H}^{(\mathrm{diss})}(\tau) = \frac{\boldsymbol{v}(\tau)}{c} \wedge \boldsymbol{E}^{(\mathrm{diss})}(\tau), \qquad \beta_+ = \frac{v(\tau+\tau_0)}{c}.$$

Le espressioni della potenza del campo elettromagnetico proprio, di reazione e dissipativo si otterranno moltiplicando i rispettivi vettori elettrici scalarmente per $\boldsymbol{v}(\tau)/\sqrt{1-\beta^2}$.

Sviluppando in serie di potenze di τ_0 le diverse formule scritte, si ottengono come primi termini non nulli quelli che ordinariamente si considerano in prima approssimazione nella teoria dell'elettrone, qualunque sia il modello che di questo si adotta.

Come applicazione è facile calcolare (nella approssimazione non relativistica) la espressione della potenza irradiata $W^{(\mathrm{diss})}$ da parte di un elettrone che si muova sotto l'azione di un campo magnetico costante H_0 (diretto lungo l'asse z). In questo caso è possibile integrare esattamente l'equazione del moto (vedi [1]).

Partendo da tali soluzioni con calcolo immediato si trova allora:

$$W^{(\mathrm{diss})}(t) = e\boldsymbol{E}^{(\mathrm{diss})}(t)\times\boldsymbol{v}(t) = -e\boldsymbol{E}^{(\mathrm{reaz})}(t)\times\boldsymbol{v}(t) = -\frac{e^2H_0^2v^2\tau_0}{2m_0c^2}\left(1+\frac{e^2\tau_0^2H_0^2}{m_0^2c^2}\right)^{-(t/\tau_0)-1},$$

che differisce dall'espressione dedotta con l'ordinario calcolo perturbativo per la presenza del termine tra parentesi, in generale di effetto trascurabile.

The Mass of a π^--Meson Determined from a σ-Star (*).

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(ricevuto il 19 Dicembre 1954)

In the routine scanning of a plate exposed to cosmic rays, a star was found which was produced by the absorption of a negative π-meson where all of the rest energy of the meson appeared to be converted into kinetic energy of charged particles. An event of this nature makes it possible to determine the rest mass of the π^--meson (1). Since such stars are extremely rare it seemed worthwhile to briefly describe this event. A photograph of the star is shown in Fig. 1. The star consists only of two prongs; a recoil track 8.6 μ long and an oppositely directed gray track. The two tracks are colinear within the limits of measurement ($\pm 4^0$), the measurement of the dip of the short recoil track being the main source of error. A track, produced by an electron of 8.5 ± 1.1 MeV is associated with the end of the recoil track (B). The energy of the electron was determined from the multiple scattering. The multiple scattering and grain density along track A establish that it was produced by a proton of 115 ± 8 MeV. The proton energy was determined from its grain density in several plates. The grain density as a function of E/E_0 was determined from π-meson tracks in the same plates.

The high proton energy and the colinearity of the two tracks suggest that only the two particles were involved in the decay. A β-emitting recoil of an appreciable range can be produced only by the absorption in the emulsion of a negative π-meson in an ^{16}O nucleus:

$$\pi^- + {}^{16}O \rightarrow p + {}^{15}C$$

$$^{15}C \rightarrow {}^{15}N + \beta^- + \nu \quad (E_{\beta_{max}} = 8.8 \text{ MeV}).$$

The absorption in heavy elements would produce a recoil of very short range. The range of the ^{15}C recoil from absorption in ^{16}O should be about 8.8 μ (2), which agrees well with the observed range of 8.6 μ.

(*) Supported in part by the Graduate School from funds supplied by the Wisconsin Alumni Research Foundation.

(1) R. C. Cornelius, C. P. Sargent, M. C. Rinehart, L. M. Lederman and K. Rogers; *Phys. Rev.*, **92**, 1583 (1953). The rest mass of a π^--meson was found from the measurement of the energy of the electron pair from the reaction $\pi^- + P \rightarrow e^- + e^+ + \gamma + N$.

(2) J. F. Miller: *UCRL Report 1902* (1952). Unpublished.

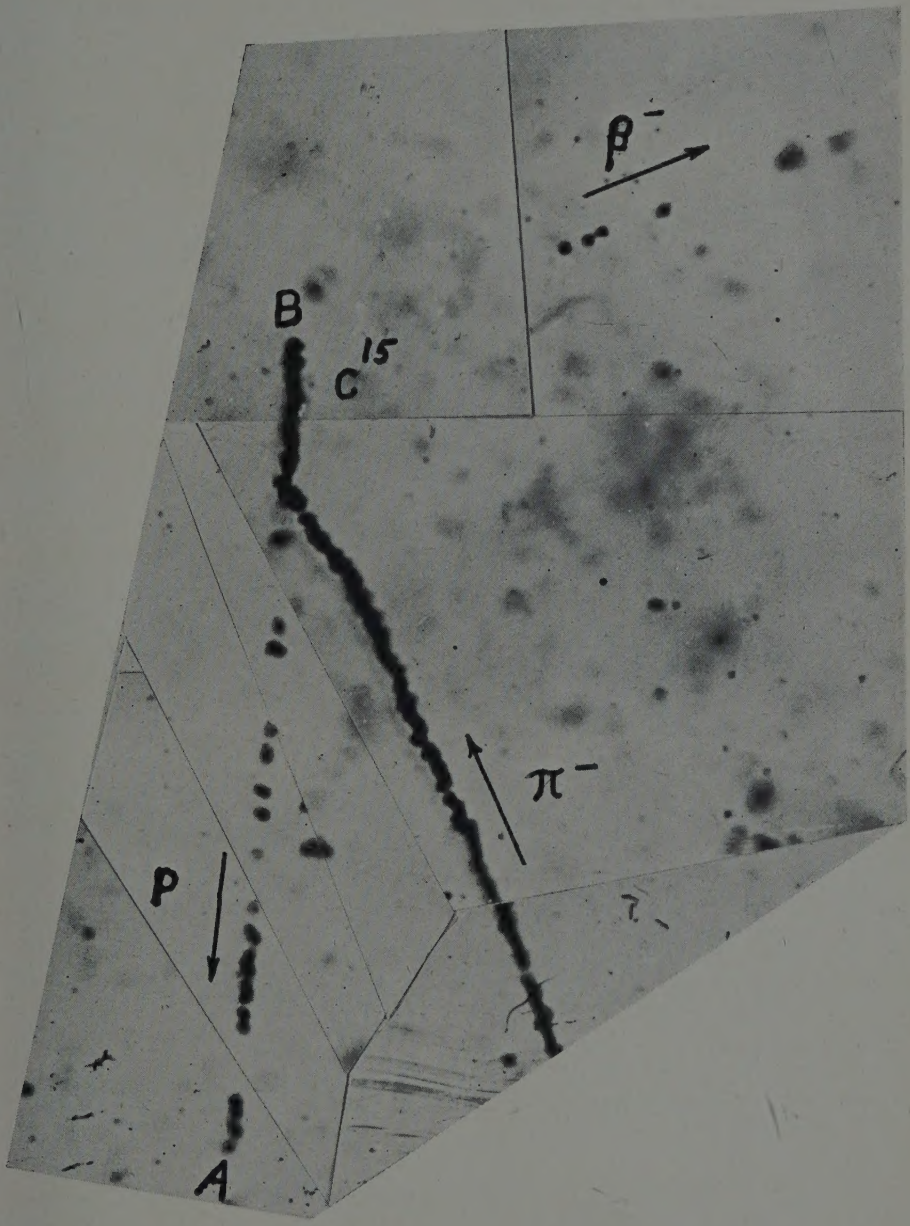

Fig. 1. – A negative π-meson star is shown in the above photograph. Track *A* was produced by a proton of 115 ± 8 MeV; track *B* was produced by a ^{15}C nucleus. The visible energy release was found to be 123 ± 8 MeV.

LIBRI RICEVUTI E RECENSIONI

J. O. HIRSCHFELDER, C. F. CURTISS and R. B. BIRD – *Molecular Theory of Gases and Liquids*. New York, John Wiley & Sons Inc., 1954, vol. di 1219 pagg., $ 20.

Questo libro esce dal Naval Research Laboratory della University of Wisconsin, che sotto la direzione del prof. HIRSCHFELDER ha molto contribuito alle nostre più recenti conoscenze in questo campo. Come dicono gli autori nell'introduzione, questo libro è destinato a diverse categorie di persone, quali gli studenti di fisica e di chimica, gli ingegneri chimici ed anche i fisici e chimici teorici. Ne è risultato un libro di dimensioni notevoli (e purtroppo di costo anche notevole), in cui troviamo capitoli introduttivi molto elementari, ed altri che sono invece rivolti solo a specialisti.

Dopo una settantina di pagine con richiami vari di meccanica classica e quantistica, di teoria dei gas e di forze intermolecolari; il libro viene diviso in tre parti: proprietà dei gas e dei liquidi nei fenomeni di equilibrio, proprietà fuori dell'equilibrio, e teoria delle forze intermolecolari. Ogni parte è ancora suddivisa in numerosi capitoli, alcuni sempre introduttivi, ed altri di carattere avanzato, in cui finalmente troviamo raccolti e trattati con ampiezza tutti quei fatti che eravamo abituati a trovare come applicazioni ed esempi nei libri di meccanica statistica e di meccanica quantistica. Questo sforzo di sintesi è spesso ostacolato dalla incompleta conoscenza che ancora possediamo di questa materia, specialmente per quanto riguarda i liquidi. Queste lacune vengono bene sottolineate dagli autori, onde il libro produce nel ricercatore una notevole azione stimolante.

Alcuni capitoli vertono su argomenti ancora in rapido sviluppo, e data la novità ci sembra bene riportare il titolo di alcuni di essi: Equazioni di stato per i gas densi ed i liquidi, Teoria quantistica ed equazioni di stato, Fenomeni di trasporto e teoria quantistica, La propagazione delle fiamme e delle detonazioni, Le forze intermolecolari tra molecole asimmetriche. Chiudono il libro una serie di tavole che riusciranno certo utili a quanti lavorano in questa branca della fisica molecolare.

G. CARERI

E. SEGRÈ – *Experimental Nuclear Physics*, vol. II, pp. VIII+660, John Wiley and Sons, Inc., New York; Chapman and Hall, Lim., London, 1953.

È questo il secondo dei tre volumi, compilati sotto la direzione di E. SEGRÈ, e aventi lo scopo dichiarato di mettere a disposizione del fisico sperimentale un chiaro e rapido riassunto di tutti i fatti

più importanti relativi ai nuclei atomici conosciuti sino al 1952, indicandone, insieme, nelle grandi linee, l'interpretazione teorica. Del primo volume dell'opera ci siamo già occupati in questa sede nel Marzo scorso (*Nuovo Cimento*. 11, 332 (1954)).

Questo secondo volume contiene due grossi articoli, il sesto della raccolta: *A Survey on nuclear reactions* di PHILIP MORRISON; e il settimo: *The Neutron* di BERNARD T. FELD.

Il primo dei due, di 207 pagine, inizia discutendo l'applicazione dei teoremi classici di conservazione alle reazioni nucleari. Introduce quindi i concetti fondamentali di sezione d'urto, densità e larghezza dei livelli, rinviando per la classificazione dei vari tipi di reazioni nucleari ad una breve appendice che contiene anche l'avviamento bibliografico fondamentale (9 pagine, bibliografia compresa).

Segue una discussione qualitativa de vari modelli nucleari e delle relazioni tra larghezza dei livelli e sezioni d'urto. Il concetto di nucleo composto serve da guida per una discussione dell'andamento generale di una reazione nucleare, dopo di che viene sviluppata con una certa ampiezza la teoria della dispersione.

Il seguito dell'articolo è dedicato ad un rapido esame di taluni tipi di reazioni nucleari. Vengono trattate successivamente le reazioni a basse energie; le reazioni iniziate da deutoni, ivi compreso lo « stripping »; i processi d'irraggiamento e le reazioni foto-nucleari; la scissione. L'ultimo capitolo, relativamente più ampio dei precedenti (51 pagine), tratta delle reazioni nucleari ad elevate energie (demolizioni ed evaporazioni) con esclusione di quelle in cui intervengono mesoni.

Nel suo complesso l'articolo di MORRISON svolge il suo argomento essenzialmente dal punto di vista teorico. Pure essendo indubbiamente assai utile e rientrando nel quadro dell'opera per le idee generali che discute ed offre allo sperimentatore, è tuttavia quasi privo di materiale sperimentale effettivamente organizzato. Molte lacune si possono giustificare pensando che i relativi argomenti vengano in seguito ripresi con maggiore ampiezza in altri articoli, ma ciò non toglie che talora resti l'impressione di una certa frettolosità nella compilazione.

Il successivo articolo di FELD, di 379 pagine, rappresenta invece esattamente ciò che desidera trovare il fisico sperimentale, aprendo una monografia riassuntiva su un argomento che lo interessa. Le idee generali e le considerazioni teoriche servono di guida per una effettiva organizzazione del materiale sperimentale disponibile, in un quadro limpido, in cui la indispensabile selezione deriva da un accurato lavoro di revisione e di critica.

L'articolo è suddiviso in cinque capitoli. Il primo di essi (39 pagine) a carattere introduttivo, espone brevemente la storia della scoperta del neutrone e discute le proprietà fondamentali e le interazioni di questa particella con altre particelle elementari (protoni, neutroni ed elettroni).

Il secondo capitolo (110 pagine) è dedicato alle interazioni del neutrone con i nuclei atomici alle varie energie, con esclusione per altro dei fenomeni in cui intervengono mesoni. La teoria delle risonanze per neutroni lenti e la dipendenza dalla energia delle sezioni d'urto sono trattate estesamente. Si esaminano, quindi, con abbondanza di dati sperimentali raccolti in numerose tabelle riassuntive, i vari tipi di reazioni nucleari iniziate da neutroni.

Il terzo capitolo (103 pagine) è dedicato alle sorgenti e ai rivelatori di neutroni di cui sono raccolti e posti a confronto tutti i dati importanti conosciuti. È poi trattata con una certa ampiezza la spettroscopia dei neutroni lenti e il problema della calibrazione delle sorgenti.

Gli ultimi due capitoli, rispettivamente di 48 e 61 pagine, sono entrambi

dedicati alle interazioni dei neutroni con la materia. Il primo tratta del rallentamento e della diffusione dei neutroni per urto e delle applicazioni della relativa teoria ai reattori nucleari. Il secondo riguarda la diffrazione dei neutroni lenti e le sue applicazioni allo studio della struttura della materia nonchè i fenomeni di polarizzazione magnetica dei neutroni.

L'articolo termina con un elenco bibliografico comprendente oltre 650 lavori.

Si può dire in conclusione che questo secondo volume dell'opera diretta da SEGRÈ non è certamente inferiore al primo e fa desiderare una sollecita pubblicazione del volume ancora mancante.

M. AGENO

L. J. BELLAMY – *The Infra-Red Spectra of Complex Molecules*; un vol. di XVII+323 p. Ed. Methuen, London 1954; 35 s.

Lo studio degli spettri di assorbimento nell'infrarosso acquista sempre maggiore importanza, non solo dal punto di vista teorico ma anche da quello della Chimica pratica, specialmente per lo studio strutturale delle molecole complesse. Per questo tipo di ricerche è necessario disporre di una grande quantità di notizie sui risultati sperimentali sino ad oggi ottenuti e sulle interpretazioni proposte, informazioni cioè che sono reperibili solo attraverso un accurato esame di molte fonti bibliografiche.

Lo scopo del lavoro del dott. BELLAMY è principalmente quello di raccogliere ordinatamente tutti i dati disponibili sull'argomento e di esporli, accompagnati da un commento e da una valutazione del grado di attendibilità che si può dare a ciascuno. Dopo una breve introduzione egli presenta quindi la materia divisa secondo la natura dei composti e dei legami chimici. Ogni capitolo reca una parte introduttiva ed un elenco bibliografico; completano il testo riproduzioni di spettri e tavole di correlazione, nelle quali, in corrispondenza di ogni composto o gruppo, viene indicata l'estensione delle relative bande di assorbimento. La consultazione del volume è facilitata, oltre che dall'indice per materie, da un indice dei composti chimici presi in considerazione. Riteniamo che il volume possa riuscire di grande utilità a chiunque si occupi praticamente di spettroscopia dell'infrarosso e possa anche fornire ai chimici un chiaro quadro riassuntivo di quali risultati ci si possano attendere da un esame spettroscopico delle sostanze studiate.

F. A. LEVI

Questo fascicolo è stato licenziato dai torchi il 28-XII-1954

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